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Limits of Spaces

Summer semester 2019 **01.07.2019**

Exercise sheet 5

Exercise 17

(Synthetic definition of a manifold - 5 points) A function $f : M \to \mathbb{R}$ on a geodesic space (M, d) is called *affine* if for all geodesics γ it holds

$$f(\gamma_t) = (1-t)f(\gamma_0) + tf(\gamma_1).$$

We say that a metric space has *finite dimensional* if all its tangent space are proper metric spaces.

Fact. *A finite dimensional geodesically complete space (M, d) is isometric to a finite dimensional normed space if for all distinct $x, y \in M$ there is an affine function $f_{x,y}$ such that $f_{x,y}(x) \neq f_{x,y}(y)$.*

Assume (M, d) is geodesically complete, has finite dimension and for all $x_0 \in M$ there is a function $C = C_{x_0} : (0, \infty) \to (0, \infty]$ with $\liminf_{\epsilon \to 0} C(\epsilon) > 0$ such that for all $x, y, z \in B_\epsilon(x_0)$ and all midpoints m of y and z and all geodesics γ and η connecting x and y and resp. x and z it holds

$$d^2(x, m) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - Cd^2(x, y)$$

and

$$d(\gamma_t, \eta_t) \geq (1 - C\epsilon^2)td(\gamma_1, \eta_1).$$

Assume for all geodesics lines γ in the tangent space it holds $b^+_\gamma + b^-_\gamma \equiv 0$. Show that (M, d) is a finite dimensional manifold.

(4 extra points) Show that the tangent spaces are non-positively curved and each two co-lines are of bounded distance. Use the exercise below and don't use the assumption "$b^+_\gamma + b^-_\gamma \equiv 0$" to prove that Busemann functions are affine.

Exercise 18

(Convex functions - 6 points) On a closed connected interval $I \subset \mathbb{R}$ we say a function $f : I \to [-\infty, \infty]$ is a non-trivial convex function if $f \not\equiv -\infty$ and for all $a, b \in I, t \in (0, 1)$ it holds $f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$.

Show that $b \mapsto \frac{f(b)-f(a)}{b-a}$ is monotone in b. Conclude that one-side derivatives $\partial^\pm f$ exist and are monotone as well.

We say an affine function $\ell_{a,b} : t \mapsto at + b$ is a subdifferential of f at t_0 if $f(t) \geq \ell(t)$ and $f(t_0) = \ell(t_0)$. Prove that the each t_0 admits a subdifferential which is unique if and only if the one-sided derivatives agree.

Show that the supergraph $\{(t, s) \mid s \geq f(t)\}$ of f is given by the intersection of the supergraphs of all subdifferentials of f. Use this to prove that f is constant if and only $\partial^\pm f$ is not constant.

(1 extra points) Show that a real-valued function on some open I is convex if and only if at each point it admits a subdifferential.

(2 extra points) Prove that a real-valued convex function on $\mathbb{R}$ is locally Lipschitz continuous.

Exercise 19

(Flat strip theorem - 2 points) Assume (M,d) is non-positively curved and $\gamma,\eta:\mathbb{R}\to M$ are two geodesic lines such that $t\mapsto d(\gamma_t,\eta_t)$ is bounded. Show that $t\mapsto d(\gamma_{t+s},\eta_t)$ is constant for all $s\in\mathbb{R}$.

(2 extra points) Choose geodesics $\zeta^{(t)}:[0,1]\to M$ connecting γ_t and η_t. Prove that $t\mapsto\zeta_s^{(t)}$ is a geodesic line for all $s\in[0,1]$.

(2 extra points) Prove that there is a normed space $(\mathbb{R}^2,d_{\|\cdot\|})$ and an isometry between $(\{\zeta_s^{(t)}\}_{s\in[0,1],t\in\mathbb{R}},d)$ and the strip $([0,d(\gamma_0,\eta_0)],\times\mathbb{R},d_{\|\cdot\|})$.

(4 extra points) Show that whenever $\xi,\rho:[0,1]\to M$ are geodesics such that $t\mapsto d(\xi_t,\rho_t)$ is affine then the union of the points on the geodesics connecting ξ_s and ρ_t, $s,t\in[0,1]$ is isometric to a convex subset of a 2-dimensional normed space.

Exercise 20

(Finite dimensionality in metric trees - 3 points) For a metric tree define $D_M(\epsilon,x)=\#\partial B_r(x)$ for $r\geq 0$ and $x\in M$.

Assume (M,d) is a geodesically complete metric tree. Show (a) (M,d) is proper if and only if D_M is real-valued and (b) (M,d) is finite dimensional if and only if D_M is a locally bounded function on $[0,\infty)\times M$. (Hint: Prove D_M is monotone.)

(2 extra points) Give an example of a proper metric tree that are neither geodesically complete and nor finite dimensional such that D_M is everywhere equal to ∞.

Deadline 15.07.2019 in mailbox C-buidling 3rd floor or office C5P08.
You reach the website of the lecture under `http://tinyurl.com/Limits-of-Spaces`.