LECTURE NOTES
„LINEAR PARTIAL DIFFERENTIAL EQUATIONS“

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Contents

Note: The following notes are written for the lecture “Linear Partial Differential Equations” during the summer semester 2017 at the University of Tübingen.

1. Classes of PDEs

Definition 1.1 (Partial Differential Equation)**.** Let $\Omega \subset \mathbb{R}^n$ be an open domain and F a function on $\mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \mathbb{R}^n \times \mathbb{R} \times \Omega$. Then an equation of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0 \quad \text{for all } x \in \Omega$$

for an unknown function $u \in C^k(\Omega)$ is called a *partial differential equation of k-th order.*

- The equation is called *linear*, if F is linear in all but the last entry. In this case the equation transformed into the following equation
$$\sum_{|\gamma|\le k} a^\gamma(x)\partial_\gamma u(x) = f(x)$$
where γ is a multi-index and $\partial_\gamma = \partial_{\gamma_1} \cdots \partial_{\gamma_l}$ where $\gamma = (\gamma_1, \ldots, \gamma_l)$.
- The equation is called *semi-linear,* if F is lienar in the first entry. In this case the equation is of the form
$$\sum_{|\gamma|=k} a^\gamma(x)\partial_\gamma u(x) + \tilde{F}(D^{k-1}u(x), \ldots, u(x), x) = 0 \quad \text{for all } x \in \Omega$$
where $\tilde{F}$ is a function on $\mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R} \times \Omega$.
- The equation is called *quasi-linear*, if there are function $a_\gamma, \tilde{F} : \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that the equation can be transformed into the following

$$\sum_{|\gamma|=k} a^\gamma(D^{k-1}u(x), \ldots, u(x), x)\partial_\gamma u(x) + \tilde{F}(D^{k-1}u(x), \ldots, u(x), x) = 0 \quad \text{for all } x \in \Omega.$$

In the course of the lecture we will focus mainly on linear partial differential equations of second order. The general equation is then given as follows: Let $\Omega \subset \mathbb{R}^n$ be an open and connected domain, $a^{i,j} : \Omega \to \mathbb{R}$, $b^k : \Omega \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}$ (continuous) functions on Ω where $i, j, k \in \{1, \ldots, n\}$. Then for a function $u \in C^2(\Omega)$ we define an operator $L : C^2(\Omega) \to C^0(\Omega)$ by

$$Lu(x) = \sum_{i,j=1}^{n} a^{i,j}(x)\partial_i\partial_j u(x) + \sum_{k=1} b^k(x)\partial_k u(x) + c(x)u(x).$$

Remark. One may verify that $Lu(x) = \tilde{L}u(x)$ for all $u \in C^2(\Omega)$ if $a^{i,j}$ is replaced by its symmetrization $a^{i,j}_{\text{sym}} = \frac{1}{2}(a^{i,j} + a^{j,i})$. Thus without loss of generality we can assume that the matrix $(a^{i,j}(x))^n_{i,j=1}$ is a symmetric matrix.

To get a partial differential *equation* one needs in addition a function $f \in C^0(\Omega)$ and asks whether $u \in C^2(\Omega)$ satisfies the equation

$$Lu = f \quad \text{in } \Omega.$$

The most natural one is to look for (non-trivial) functions u such that $Lu = 0$. As L is linear one can also ask whether there are (non-trivial) u and a $\lambda \in \mathbb{R}$ such that

$$Lu = \lambda u \quad \text{in } \Omega$$

in which case we call u an eigenfunction of L. Instead of solving $Lu = \lambda u$ one may equivalently solve the equation $L_\lambda u = 0$ where $L_\lambda = L - \lambda\,\text{id}$.

Example 1.2 (Classical Examples)**.** Classical linear PDEs are given as follows

Laplace equation: Let $L = \Delta := \sum_{i=1}^{n} \partial_{ii}$ then

$$\Delta u = 0 \quad \text{in } \Omega$$

is called the Laplace equation and solutions u are called harmonic function.

Poisson equation: More generally, one may look the *Poisson equation*

$$\Delta u = f \quad \text{in } \Omega$$

Heat equation: Assume $\Omega_T \subset \mathbb{R} \times \mathbb{R}^n$ where the first coordinate will be denoted by t and Δ is defined as above. Then the following is called the *wave equation*

$$\partial_t u - \Delta u = 0 \quad \text{in } \Omega_T$$

Wave equation: As above $\Omega_T \subset \mathbb{R} \times \mathbb{R}$:

$$\partial_{tt} u - \Delta u = 0 \quad \text{in } \Omega_T$$

A class of examples of non-linear PDEs a

Non-linear Laplace equation:

$$\Delta u = f(u) \quad \text{in } \Omega$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function. This is an example of a quasi-linear PDE.

p-Laplace equation:

$$\mathsf{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \Omega.$$

This is an example of a semi-linear PDE.

Minimal surface equation:

$$\mathsf{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \Omega.$$

Monge-Ampère equation:

$$\mathsf{det} \left(D^2 u \right) = 0 \quad \text{in } \Omega.$$

Note that just asking for $Lu = 0$ gives in general an underdetermined system if $\partial\Omega$ is non-empty. This is best observed in the one-dimensional setting.

Example (One dimensional)**.** In case $n = 1$, $\Omega = (a, b)$ for $a < b \in \mathbb{R}$ and the operator L is of the form

$$Lu = au'' + bu' + cu$$

for functions $a, b, c : (a, b) \to \mathbb{R}$. Hence the linear partial differential equation of second order

$$Lu = f$$

is just a linear ordinary differential equation of second order. For $a \equiv 1$ and $b \equiv c \equiv f \equiv 0$ the equation

$$u'' = 0$$

is solved by all affine functions, i.e. for $a_0, b_0 \in \mathbb{R}$

$$u = a_0 x + b_0$$

satisfies $u'' = 0$. Note that this is obviously underdetermined so that it is natural to find solution with given boundary values, i.e. find u such that $u(a) = g(a)$ and $u(b) = g(b)$ with for some $g : \{a, b\} \to \mathbb{R}$, or shorter written as $u\big|_{\partial\Omega} = g$.

Thus also in the higher dimensional setting it is natural to ask for solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ that satisfy

$$\begin{cases} Lu = f & \text{in } \Omega \\ u\big|_{\partial\Omega} = g. \end{cases}$$

A solution of a PDE with boundary data g is usually called a solution to the *Dirichlet problem* (with boundary data g).

Remark. By linearity it is possible to focus only on case where either $f \equiv 0$ or $g \equiv 0$ (given that there are sufficiently many solutions $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Lu = f$ and resp. $u\big|_{\partial\Omega} = 0$:

(1) If $\tilde{u} \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $L\tilde{u} = f$ and $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solves $Lu = 0$ with boundary data $\tilde{g} = g - \tilde{u}\big|_{\partial\Omega}$ then $u = v + \tilde{u}$ solves $Lu = f$ with boundary data g..

(2) If $\hat{u} \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\hat{u}\big|_{\partial\Omega} = g$ and $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solves $Lu = \tilde{f}$ with boundary data $g \equiv 0$ where $\tilde{f} = f - L\hat{u}$ then $u = v + \tilde{u}$ solves $Lu = f$ with boundary data g.

In general the equation $Lu = 0$ with given boundary data might still be unsolvable if the highest order coefficients are too general. As we can assume the matrix $(a^{i,j})_{i,j=1}^n$ is symmetric, it can be diagonalized so that we can define the following three main categories of PDEs:

Elliptic: The operator L is called *elliptic* if for all $x \in \Omega$ the matrices $(a^{i,j}(x))_{i,j=1}^n$ have positive eigenvalues[1]. This can be expressed by assuming there are functions $\lambda, \Lambda : \Omega \to (0, \infty)$ such that

$$\lambda(x) \sum_{i=1}^n \xi_i \xi_i \le \sum_{i=1}^n a^{i,j}(x) \xi_i \xi_j \le \Lambda(x) \sum_{i=1}^n \xi_i \xi_i$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$ and all $x \in \Omega$. If λ and Λ can be chosen independently of $x \in \Omega$ then we say the operator is *uniformly elliptic*.

Hyperbolic: The operator L is called hyperbolic if for all $x \in \Omega$ the matrices $(a^{i,j}(x))_{i,j=1}^n$ non-zero eigenvalues. Note that if $x \mapsto a^{i,j}(x)$ is continuous then the number of positive (and resp. negative) eigenvalues remains constant along Ω by connectedness of Ω and continuity of the spectrum of the matrices $(a^{i,j}(x))_{i,j=1}^n$. In a simpler setting, one only looks at the class of hyperbolic PDEs where $a^{1,1}(x) = -1$, $a^{1,j}(x) = a^{j,1}(x) = 0$ such that $(a^{i,j}(x))_{i,j=2}^n$ is elliptic and independent of the first coordinate. In that case a PDE of the form $\partial_{tt} - Lu = f$ in a domain $\Omega_T \subset \mathbb{R} \times \mathbb{R}^n$ where L is an elliptic operator L.

Similarly one could look at operators Lu on domains $\Omega_T \subset \mathbb{R} \times \mathbb{R}^n$ where $a^{i,j}$, b^k and c depend on the first coordinate t. Note, however, that an equation

$$Lu = f$$

would be just a time-dependent elliptic equation. An alternative is to add a derivative $\partial_t u$ in the equation and obtain the following:

[1] resp. negative eigenvalues as we might switch signs and replace L by $-L$.

Parabolic: A parabolic PDE on the domain $\Omega_T \subset \mathbb{R}\times\mathbb{R}^n$ is of the form $\partial_t - Lu = 0$ where L (resp. it's coefficients $a^{i,j}$, b^k and c) are allowed to depend on t, in which case it would be call a time-dependent parabolic PDE.

The focus of the lecture is on elliptic and parabolic PDEs as many techniques are similar.

2. Harmonic functions on $\mathbb{R}^n$

In this section we study harmonic functions on domain $\Omega \subset \mathbb{R}$.

Definition 2.1 (Harmonic function)**.** A function $u \in C^2(\Omega)$ is called *subharmonic* if $\Delta u \geq 0$ and it is called *superharmonic* if $\Delta u \leq 0$. If it is both sub- and superharmonic then we say u is *harmonic*.

Observe that the sum of two subharmonic functions is itself subharmonic.

2.1. The mean value property, maximum principle and Harnack's inequality.

In the following we frequently use the notation $\fint_A f d\mu = \frac{1}{\mu(A)}\int_A f d\mu$ for a measure μ. In case μ is the Lebesgue measuren then this is written as $\fint_A f dx = \frac{1}{\lambda^n(A)}\int_A f dx$.

Theorem 2.2 (Mean value property)**.** *Assume $u \in C^2(\Omega)$ is subharmonic. Then for all $B_r(x) \subset\subset \Omega$ it holds*

$$u(x) \leq \fint_{\partial B_r(x)} u(z)dz$$

and

$$u(x) \leq \fint_{B_r(x)} u(y)dy.$$

Remark. Later on we show that the mean value property is actually equivalent to being subharmonic for all C^2-functions.

Proof. In polar coordinates it holds

$$\fint_{\partial B_r(x)} u(z)dz = r^{n-1}\int_{\mathbb{S}^{n-1}} u(x + r\omega)d\omega.$$

Also note that for $z = r\omega \in \partial B_r(x)$ unit outer normal ν is given by ω. Hence

$$\partial_\nu u(z) = \frac{d}{ds}u(x+s\omega)\bigg|_{s=r}.$$

Then polar coordinates show

$$\int_{\partial B_r(x)} \partial_\nu u(z)dz = r^{n-1}\int_{\mathbb{S}^{n-1}} \frac{d}{ds}u(x+s\omega)\bigg|_{s=r} d\omega.$$

Because $u \in C^1(\Omega)$ we can pull out derivative under the integral and obtain

$$\begin{aligned}\int_{\partial B_r(x)} \partial_\nu u(z)dz &= r^{n-1}\frac{d}{ds}\left(\int_{\mathbb{S}^{n-1}} u(x+s\omega)d\omega\right)\bigg|_{s=r}. \\ &= r^{n-1}n\omega_n\frac{d}{ds}\left(\fint_{\partial B_s(x)} u(z)dz\right)\bigg|_{s=r}\end{aligned}$$

where we used again the polar coordinate transformation and the fact that $|\partial B_r| = n\omega_n r^{n-1}$.

Now Green's formula applied yields

$$\begin{aligned} 0 &\leq \int_{B_r(x)} \Delta u(y)dy. \\ &= \int_{\partial B_r(x)} \partial_\nu u(z)dz \\ &= r^{n-1} n\omega_n \frac{d}{ds}\left(⨍_{\partial B_s(x)} u(z)dz \right)\Big|_{s=r} \end{aligned}$$

implying that

$$r \mapsto ⨍_{\partial B_r(x)} u(z)dz$$

is non-decreasing in $r \mapsto (0, r_0)$ for some $r_0 > r$ with $B_{r_0}(x) \subset \Omega$. Since u is continuous at x it holds

$$u(x) = \lim_{r\to 0} ⨍_{\partial B_r(x)} u(z)dz.$$

This proves the first claim. To obtain the second claim note that

$$\omega_n r^n u(x) = \int_0^r n\omega_n s^{n-1} u(x)ds = \int_0^r \int_{\partial B_s(x)} u(z)dzds = \int_{B_r(x)} u(y)dz.$$

Because $|B_r(x)| = \omega_n r^n$ we obtain the second claim. □

Corollary 2.3 (Strong Maximum Principle)**.** *For all subharmonic functions $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ it holds*

$$\sup_\Omega u = \sup_{\partial\Omega} u$$

and if for some $x_0 \in \Omega$ it holds

$$u(x_0) = \sup_\Omega u$$

then u is constant on Ω. In particular, if $u \geq 0$ and $u\big|_{\partial\Omega} = 0$ then $u \equiv 0$.

Proof. Let $x_0 \in \bar{\Omega}$ be such that $u(x_0) = \sup_\Omega u$. If $x_0 \in \partial\Omega$ there is nothing to prove. In case $x_0 \in \Omega$ there is a ball $B_r(x_0) \subset\subset \Omega$ such that the mean value property holds for $B_r(x_0)$. It suffices to show that u is constant.

Now the choice of x_0 yields the following

$$\sup_\Omega u = u(x_0) \leq ⨍_{B_r(x_0)} u(y)dy \leq \sup_\Omega u.$$

This, however, can only hold if $u(y) = u(x_0)$ for all $y \in B_r(x_0)$ implying that u is locally constant. As Ω is connected we see that u must be constant. □

Because $\pm u$ is subharmonic for each harmonic function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ we obtain the following corollary which gives also a uniqueness result.

Corollary 2.4. *For all harmonic functions $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ it holds*

$$\sup_\Omega |u| = \sup_{\partial\Omega} |u|.$$

In particular, if $u_1, u_2 \in C^2(\Omega) \cap C^0(\bar{\Omega})$ are harmonic in Ω with $u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega}$ then $u_1 \equiv u_2$.

Proposition 2.5. *Let $u \in C^0(\Omega)$. Then the following are equivalent.*

- *for all $B_r(x) \subset\subset \Omega$ it holds $u(x) \le \fint_{\partial B_r(x)} u(z)dz$*
- *for all $B_r(x) \subset\subset \Omega$ it holds $u(x) \le \fint_{B_r(x)} u(y)dy$*
- *for all $B_r(x) \subset\subset \Omega$ it holds $u(x) \le h(x)$ where h is a harmonic function on $B_r(x)$ with boundary data $g \ge u\big|_{\partial B_r(x)}$.*

If, in addition, $u \in C^2(\Omega)$ then either of the condition above is equivalent to u being subharmonic.

Proof. Exactly as in the proof of the mean value property, the first property implies the second by integration.

Assume the second property holds: then the maximum principle holds on $B_r(x)$ for $u-h$ whenever h is a harmonic function on $B_r(x)$ with boundary data $u\big|_{\partial B_r(x)}$. Thus

$$\sup_{B_r(x)} u - h = \sup_{\partial B_r(x)} u - h \le 0$$

implying that $u \le h$.

Let[2] h is a harmonic function on $B_r(x)$ with boundary data $u\big|_{\partial B_r(x)}$. Then the mean value property holds for h. Thus assuming the third property we get

$$u(x) \le h(x) = \fint_{\partial B_r(x)} h(z)dz = \fint_{\partial B_r(x)} u(z)dz.$$

It remains to show that either of the first three properties implies that u is subharmonic if $u \in C^2(\Omega)$. Assume by contradiction $\Delta u(x) < 0$ for some $x \in \Omega$. Then there is an open ball $B_r(x) \subset\subset \Omega$ such that $\Delta u(x) \le -\epsilon < 0$ for all $y \in B_r(x)$. In that case u is superharmonic in $B_r(x)$. However, this implies that

$$u(y) \ge \fint_{B_s(y)} udy'$$

for all $B_s(y) \subset\subset B_r(x)$ (actually for all $B_s(y) \subset B_r(x)$ since $u \in C^0(\bar{B}_r(x))$). However, this means

$$u(y) = \fint_{B_s(y)} udy'.$$

Pick $y \in B_r(x)$ and define a function $v_\epsilon \in C^2(\bar{B}_r(x))$ by

$$v_\epsilon(y) := u(y) + \frac{\epsilon}{n}\|y - x\|^2_{\text{Euclid}}$$

and note that $v_\epsilon(y) = u(y)$. Since $\Delta u \le -\epsilon$ on $B_r(x)$ we have

$$\Delta v_\epsilon \le -\epsilon + \frac{\epsilon}{n}\Delta\|y - x\|^2_{\text{Euclid}} \le 0,$$

i.e. v_ϵ is superharmonic on $B_r(x)$. In particular, it holds

$$v_\epsilon(y) \ge \fint_{B_s(y)} v_\epsilon dy'$$

[2]Strictly speaking we assume the existence of harmonic functions in balls with given boundary data, see next section.

for all $B_s(y) \subset\subset B_r(x)$. However, this leads to the following contradiction

$$0 < \fint_{B_s(y)} \frac{\epsilon}{n}\|y' - x\|^2_{\text{Euclid}} dy'$$
$$= \fint_{B_s(y)} v_\epsilon - u dy' \le v_\epsilon(y) - u(y) = 0.$$

□

The proposition shows allows us to define a weak form of (sub-/super-)harmonicity.

Definition 2.6 (Mean-value harmonic)**.** A (bounded) measurable function $u \in C^0(\Omega)$ is called *mean-value subharmonic* if for all $B_r(x) \subset\subset \Omega$

$$u(x) \le \fint_{B_r(y)} u(y)dy.$$

It is *mean-value superharmonic* if $-u$ is mean-value subharmonic. If it is both mean-value sub- and superharmonic we call it *mean-value harmonic.*

Remark. Later we prove that a mean-value harmonic function is indeed a C^2-function and hence harmonic.

The following is now a consequence of monotonicity of the averaged integral over balls.

Lemma 2.7. *Let $\{u_i\}_{i\in I}$ be mean-value subharmonic functions in Ω for some finite index set I and define*

$$u(x) := \sup_{i\in I} u_i(x).$$

Then u is mean-value subharmonic.

Theorem 2.8 (Harnack's Inequality)**.** *Assume $u \in C^2(\Omega)$ is a non-negative harmonic function in Ω. Then for each $\Omega' \subset\subset \Omega$ there a constant $C = C(n, \Omega', \Omega)$ such that*

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u.$$

Proof. Let $B_{4r}(y) \subset\subset \Omega$ then for all $x_1, x_2 \in B_r(y)$ it holds

$$u(x_1) \le \fint_{B_r(x_1)} u(y)dy \le \frac{1}{|B_r(x_1)|} \int_{B_{2r}(y)} u(y)dy$$

and

$$u(x_2) \ge \fint_{B_{3r}(x_1)} u(y)dy \ge \frac{1}{|B_{3r}(x_2)|} \int_{B_{2r}(y)} u(y)dy.$$

Since $|B_{3r}(x_2)| = 3^n|B_r(x_1)|$ this shows

$$u(x_1) \le 3^n u(x_2).$$

In particular,

$$\sup_{B_r(y)} u \le 3^n \inf_{B_r(y)} u.$$

Let $\Omega' \subset\subset \Omega$ then there is a radius $r > 0$ such that for all $y \in \operatorname{cl}\Omega'$ it holds $B_{4r}(y) \subset\subset \Omega$. Since $\operatorname{cl}\Omega'$ is compact there are finitely many $y_1, \ldots, y_N$ such that

$$\Omega' \subset \bigcup_{i=1}^N B_r(y).$$

Thus for all $x_1, x_2 \in \Omega'$ there is a sequence $z_0 = x_1, z_1, \dots, z_N = x_2 \in \Omega'$ such that for each $i = 1, \dots, N$ there is an index $j_i \in \{1, \dots, N\}$ such that $z_{i-1}, z_i \in B_r(y_{j_i})$. Thus we obtain

$$u(z_0) \le 3^n u(z_1) \le 3^{2n} u(z_2) \le \dots \le 3^{Nn} u(z_N)$$

showing that

$$\sup_{\Omega'} u \le 3^{Nn} \inf_{\Omega'} u.$$

□

Remark. The Harnack inequality also implies the strong maximum principle, indeed, if $u \in C^0(\bar{\Omega})$ and $M = \sup_\Omega u = u(x_0)$ for some $x_0 \in M$ then $M - u \ge 0$ is a non-negative harmonic function. Thus for all $\Omega' \subset\subset \Omega$ with $x_0 \in \Omega'$ it holds

$$\sup_{\Omega}(M - u) \le \inf_{\Omega'}(M - u) = 0$$

implying $u \equiv M$.

2.2. **Poisson's formula for solutions on the ball.**

Theorem 2.9 (Poisson's formula)**.** *Let $g \in C^0(\partial B_1(0))$ be a continuous function and define a function $u : \bar{B}_1(0) \to \mathbb{R}$ as follows*

$$u(x) = \begin{cases} \frac{1-|x|^2}{n\omega_n} \int_{\partial B_1(0)} \frac{g(z)}{|x-z|^2} dz & x \in B_1(0) \\ g(x) & x \in \partial B_1(0). \end{cases}$$

Then $u \in C^\infty(B_1(0)) \cap C^0(\bar{B}_1(0))$ is a harmonic function in $B_1(0)$ with boundary data g.

Proof. Exercise. □

Corollary 2.10. *Let $g \in C^0(\partial B_r(a))$ be a continuous function and define a function $u : \bar{B}_r(a) \to \mathbb{R}$ as follows*

$$u(x) = \begin{cases} \frac{r^2-|x-a|^2}{rn\omega_n} \int_{\partial B_r(a)} \frac{g(z)}{|x-z|^2} dz & x \in B_r(a) \\ g(x) & x \in \partial B_r(a). \end{cases}$$

Then $u \in C^\infty(B_r(a)) \cap C^0(\bar{B}_r(a))$ is a harmonic function in $B_r(a)$ with boundary data g.

This gives us immediately a regularity theorem for (mean value) harmonic functions in general domains.

Corollary 2.11. *If $u \in C^0(\Omega)$ satisfies the mean value property on all ball $B_r(x) \subset\subset \Omega$, i.e. $u(x) = \fint_{B_r(x)} u(y)dy$, then $u \in C^\infty(\Omega)$ and u is harmonic in Ω.*

Proof. It suffices that $u \in C^\infty(B_r(x))$ for all $B_r(x) \subset\subset \Omega$. Now let v be the harmonic function on $B_r(x)$ given by Poisson's formula with boundary data $u\big|_{\partial B_r(x)}$. Then $u-v$ is a still satisfies the mean value property function on all balls $B_s(x') \subset\subset B_r(x)$ and hence the maximum principle on $B_r(x)$. However, $(u - v)\big|_{\partial B_r(x)} \equiv 0$ implying

$$\sup_{B_r(x)} |u - v| = \sup_{\partial B_r(x)} |u - v| = 0,$$

i.e. $u \equiv v$ on $B_r(x)$. As $v \in C^\infty(B_r(x))$ this yields the claim. □

Combined with Proposition 2.5 applied to $\pm u$ we get t he following.

Corollary 2.12. *Let $u \in C^0(\Omega)$. Then the following are equivalent.*

- *for all $B_r(x) \subset\subset \Omega$ it holds $u(x) = \fint_{\partial B_r(x)} u(z)dz$*
- *for all $B_r(x) \subset\subset \Omega$ it holds $u(x) = \fint_{B_r(x)} u(y)dy$*
- *for all $B_r(x) \subset\subset \Omega$ it holds $u(x) = h(x)$ where h is a harmonic function on $B_r(x)$ with boundary data $u\big|_{\partial B_r(x)}$.*
- *u is twice differentiable in Ω (i.e. $u \in C^2(\Omega)$) and harmonic in Ω (i.e. $\Delta u = 0$)*
- *u is infinitely many times differentiable in Ω and harmonic in Ω.*

2.3. **Convergence theorems for harmonic functions.** The first lemma is just a consequence that convergence of second derivative implies convergence of the Laplacian hence being harmonic is preserved under this strong form of convergence.

Lemma 2.13. *If $u_n \in C^2(\Omega)$ is a sequence of subharmonic functions that converges locally uniformly in C^2 to a function $u \in C^2(\Omega)$ then u is subharmonic.*

Using Poisson's formula this can be improved as follows:

Proposition 2.14. *Assume $u_n \in C^2(\Omega)$ is a sequence of harmonic functions converging locally uniformly (in C^0) to a function $u \in C^0(\Omega)$ then $u \in C^\infty(\Omega)$ is harmonic.*

Proof. Note that the mean value property is preserved under uniform convergence. Thus the claim follows from Corollary 2.11. □

A slightly weaker (though also different) convergence result was obtained by Harnack using the Harnack inequality.

Theorem 2.15 (Harnack's Convergence Theorem)**.** *Let $u_n \in C^2(\Omega)$ be a sequence of harmonic functions such that $u_n \le u_{n+1}$. If for some $y \in \Omega$ the sequence $\{u_n(y)\}_{n\in\mathbb{N}}$ is bounded then $(u_n)_{n\in\mathbb{N}}$ converges locally uniformly to a harmonic function $u \in C^2(\Omega)$.*

Proof. Since $(u_n(y))_{n\in\mathbb{N}}$ is a bounded, non-decreasing sequence it is, in particular, convergence and hence a Cauchy sequence. Thus for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $m \ge n \ge N$ it holds

$$0 \le u_m(y) - u_n(y) \le \epsilon.$$

Let $\Omega' \subset\subset \Omega$ and $C = C(n, \Omega', \Omega)$ be the constant given in Theorem 2.8. Since $u_m - u_n \ge 0$ is harmonic for all $m \ge n \ge N$ we get

$$\sup_{\Omega} |u_m - u_n| = \sup_{\Omega}(u_m - u_n) \le C \inf(u_m - u_n) \le C(u_m(y) - u_m(y)) \le C\epsilon$$

showing that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0(\Omega')$. In particular, $u_n \to u$ uniformly on Ω' where $u(x) := \lim_{n\in\mathbb{N}} u_n(x)$. □

Going back to Poisson's formula we observe the following: for all $x \in B_s(a) \subset\subset B_r(a)$ the function $z \mapsto \frac{1}{|x-z|}$ is uniformly bounded (by $C_{s,r}$). Thus it suffices to assume $g \in L^1(\partial B_r(a))$ to obtain a function $u \in C^\infty(B_r(a))$ which is harmonic in $B_r(a)$. In that case we still say u is the (unique) harmonic function with boundary data g. Furthermore, the uniform convergence on the boundary data can be replace by

Proposition 2.16. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^\infty(\partial B_r(a))$ converging in L^1 to a function g. If u_n is the (unique) harmonic function with boundary data g_n and u the one corresponding to g. Then the sequence $(u_n)_{n\in\infty}$ converges locally uniformly to u in $B_r(a)$.*

Proof. Let $x \in B_s(a)$ then

$$|u(x) - u_n(x)| \le rC_{r,s} \int_{\partial B_r(a)} |g(z) - g_n(z)| dz \to 0$$

where the convergence of the right hand side does not depend on x. Hence $u_n \to u$ uniformly on $B_s(a)$ which implies the claim as $s < r$ can be chosen arbitrary. □

Corollary 2.17. *If $(g_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions in $L^\infty(\partial B_r(a))$ such that $|g_n| \le C$. Then u_n converges locally uniformly to a harmonic function $u \in C^\infty(B_r(a))$.*

Proof. Let $g = \lim g_n$ then by the Monotone Convergence Theorem it holds

$$\int_{\partial B_r(a)} |g - g_n| dz = \int_{\partial B_r(a)} g - g_n dz \to 0,$$

i.e. $g_n \to g$ in $L^1(\Omega)$. Then the previous proposition yields the claim. □

2.4. **Gradient estimates.** Note by linearity if $u \in C^2(\Omega)$ is harmonic in Ω then for $B_r(x) \subset\subset \Omega$ it holds

$$\partial_i u(x) = \partial_i \left(\fint_{B_r(x)} u dz \right) = \fint_{B_r(x)} \partial_i u dz.$$

In particular, $\partial_i u$ is harmonic in Ω for all $i = 1, \dots, n$. Furthermore, it holds

$$\nabla u(x) = \fint_{B_r(x)} \nabla u(z) dz = \frac{n}{r} \fint_{\partial B_r(x)} u \cdot \nu \le \frac{n}{r} \sup_{\partial B_r(x)} |u|.$$

Thus

$$|\nabla u|(x) \le \frac{n}{d_x} \sup_{\Omega} |u|$$

where

$$d_x = d(x, \partial\Omega) = \inf\{d(x,y) \,|\, y \in \partial\Omega\}.$$

Lemma 2.18. *If u is harmonic bounded from above and below by a constant D then for each $\Omega' \subset\subset \Omega$ there is a constant $C = C(n, D \cdot d(\Omega', \Omega))$ such that u is Lipschitz continuous on $\bar{\Omega}'$ with Lipschitz constant bounded by C.*

Combining Arzela–Ascoli and Proposition 2.14 gives the following corollary.

Corollary 2.19. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of C^2-harmonic functions in Ω that is uniformly bounded above and below by a constant D then there is a subsequence $(u_{n_k})_{n_k\in\mathbb{N}}$ such that u_{n_k} converges locally uniformly in $C^0(\Omega)$ to a harmonic function $u \in C^\infty(\Omega)$.*

Remark. If $(u_n)_{n\in\mathbb{N}}$ is non-decreasing and bounded then the sequence converges uniformly and gives yet another proof of Harnack's Convergence Theorem (Theorem 2.15).

An iterated argument also shows that $\partial_\gamma u$ is harmonic for any multi-index γ. Then as above we obtain

$$|D^k u|(x) \leq C \sup_\Omega |u| \quad x \in \Omega' \subset\subset \Omega$$

where $C = C(k, n, \Omega', \Omega)$. Looking at $(\partial_\gamma u)_{|\gamma|<k}$ would show that a bounded sequence of harmonic functions is actually C^k-compact in any $\Omega' \subset\subset \Omega$.

2.5. **Constructing harmonic functions in general domains.** The Poisson integral formula shows that it is possible to solve the Dirichlet problem on any ball with any given continuous (resp. L^1) data. In this section we use this information to construct harmonic functions on more general domains Ω and show that they satisfy certain boundary regularity if $\partial\Omega$ is not too bad.

Perron's method of subharmonic functions. Let Ω be an open, bounded and connected domain. Given $g \in C^0(\partial\Omega)$ we define the following two sets

$$S_- = S_-(g, \Omega) = \{u \in C^0(\bar\Omega) \,|\, u \text{ is mean-value subharmonic in } \Omega \text{ and } u\big|_{\partial\Omega} \leq g\}$$
$$S_+ = S_+(g, \Omega) = \{u \in C^0(\bar\Omega) \,|\, u \text{ is mean-value superharmonic in } \Omega \text{ and } u\big|_{\partial\Omega} \geq g\}.$$

Note that any function $u \in S_- \cap S_+ \subset C^0(\bar\Omega)$ would be mean-value harmonic in Ω (hence harmonic in Ω) and satisfies $u\big|_{\partial\Omega} = g$ and by uniqueness $S_- \cap S_+$ contains at most one element.

Also note that for any $u_\pm \in S_\pm$ the function $u_- - u_+$ is mean-value subharmonic with non-positive boundary data hence $u_- \leq u_+$. Since the function $v_- \equiv \inf g$ is in S_- and the function $v_+ \equiv \sup g$ is in S_+ we also have

$$\inf_{\partial\Omega} g \leq u_- \leq u_+ \leq \sup_{\partial\Omega} g$$

for all $u_\pm \in S_\pm$.

Thus it is natural to look at the following

$$u_*(x) = \sup_{u_- \in S_-} u_-(x) \in [\inf_{\partial\Omega} g, \sup_{\partial\Omega} g]$$
$$u^*(x) = \inf_{u_+ \in S_+} u_+(x) \in [\inf_{\partial\Omega} g, \sup_{\partial\Omega} g].$$

Perron's Method is to show the following:

Step 1: Show that u_* (and thus also u^*) is harmonic.
Step 2: Show $u_*\big|_{\partial\Omega} = g$ (and thus $u_* = u^*$).

Before we start with the **Step 1** we shows that a function stays mean-value subharmonic if we replace it locally a harmonic function of given boundary data.

Lemma 2.20 (Replacement Lemma)**.** *Let $u \in C^0(\bar\Omega)$ be a mean-value subharmonic function in Ω and $B_r(x) \subset\subset \Omega$. If h is a harmonic function in $B_r(x)$ with boundary data $u\big|_{\partial\Omega}$ then the following function*

$$\tilde u(x) = \begin{cases} h(x) & x \in \bar B_r(x) \\ u(x) & x \in \Omega \backslash \bar B_r(x). \end{cases}$$

Is continuous in $\bar\Omega$ and mean-value subharmonic in Ω.

Proof. Let $B_s(y) \subset\subset \Omega$ be an arbitrary ball and h be a harmonic function with $\hat{h}\big|_{\partial B_s(y)} = \tilde{u}\big|_{\partial B_s(y)}$. We need to show $\hat{h} \geq \tilde{u}$ on $\bar{B}_s(y)$.

Since u is continuous and $\tilde{u} \geq u$, it holds $\hat{h} \geq u$ on $\bar{B}_s(y)$ by subharmonicity of u. Thus it suffices to show $\hat{h} \geq h$ on $B_s(x) \cap B_r(x)$.

Note that $\partial(B_s(y) \cap B_r(x)) = \partial B_s(y) \cap \bar{B}_r(x) \cup \partial B_r(x) \cap \bar{B}_s(y)$. On $\partial B_s(y) \cap \bar{B}_r(x)$ it holds $\hat{h} = \tilde{u} = h$. And if $\tilde{y} \in \partial B_r(x) \cap \bar{B}_s(y)$ then $h(\tilde{y}) = u(\tilde{y})$. Since $u - \hat{h} \leq 0$ on $\bar{B}_s(y)$ we have by the maximum principle

$$\begin{aligned}
\sup_{\bar{B}_s(y) \cap \bar{B}_r(x)} h - \hat{h} &= \sup_{\partial(B_s(y) \cap B_r(x))} h - \hat{h} \\
&= \max\left\{ \sup_{\partial B_s(y) \cap \bar{B}_r(x)} (h - \hat{h}), \sup_{\partial B_r(x) \cap \bar{B}_s(y)} (h - \hat{h}) \right\} \\
&= \max\left\{ \sup_{\partial B_s(y) \cap \bar{B}_r(x)} (\hat{h} - \hat{h}), \sup_{\partial B_r(x) \cap \bar{B}_s(y)} (u - \hat{h}) \right\} \leq 0
\end{aligned}$$

implying $\hat{h} \geq h$ on $\bar{B}_s(y) \cap \bar{B}_r(x)$ and thus $\hat{h} \geq \tilde{u}$ on $\bar{B}_s(x)$. □

Theorem 2.21 (Perron's Method)**.** *The functions u_* and u^* are harmonic in Ω.*

Proof. It suffices to show that u_* is harmonic on each $B_r(x) \subset\subset \Omega$. Let $v_n \in S_-$ be such that

$$u_*(x) = \lim_{n \to \infty} v_n.$$

Then

$$v_n^{'} = \max\{v_1, \ldots, v_n\}$$

is also a sequence in S_- with $u_*(x) = \lim_{n\to\infty} v_n^{'}$. Using Poisson's integral formula we find harmonic functions $h_n \in C^\infty(B_r(x)) \cap C^0(\bar{B}_r(x))$ with $h_n\big|_{\partial B_r(x)} = v_n^{'}\big|_{\partial B_r(x)}$.

The previous lemma shows that the functions $\tilde{v}_n : \Omega \to \mathbb{R}$ defined by

$$\tilde{v}_n(y) = \begin{cases} h_n(y) & y \in \bar{B}_r(x) \\ v_n^{'}(y) & y \in \Omega \backslash \bar{B}_r(x) \end{cases}$$

are subharmonic. By definition $\tilde{v}_n = v_n^{'}$ on $\partial\Omega$. Thus $\tilde{v}_n \in S_-$. In particular, $\tilde{v}_n \leq u_*$. Furthermore, $(\tilde{v}_n)_{n\in\mathbb{N}}$ is still non-decreasing and bounded by $\sup_{\partial\Omega} g$ so that Harnack's Convergence Theorem (Theorem 2.15) implies[3] that on $B_r(x)$ the sequence $\tilde{v}_n\big|_{B_r(x)}$ converges uniformly to some harmonic function $\tilde{h}$ on $B_r(x)$ with $\tilde{h}(x) = u_*(x)$.

We claim $\tilde{h} = u_*$ on $B_r(x)$. If this was not the case then there is an $z \in B_r(x)$ and subharmonic function $w_n \in S_-$ with $u_*(z) = \lim_{n\to\infty} w_n(z) > \tilde{h}(z)$. As above observe that

$$v_n^{'} \leq w_n^{'} = \max\{v_n^{'}, w_n\} \in S_-$$

so that as above we obtain a harmonic function $\hat{h}$ on $B_r(x)$ with $\tilde{h} \leq \hat{h}$ and $\hat{h}(z) = u_*(z)$. Note that this also shows $\hat{h}(x) = \tilde{h}(x) = u_*(x)$. Now the strong maximum principle on $B_r(x)$ applied to $\tilde{h} - \hat{h} \leq 0$ with $\tilde{h}(x) - \hat{h}(x) = 0$ shows $\tilde{h} - \hat{h} \equiv 0$ in $B_r(x)$. However, this is a contradiction because by construction $\hat{h}(z) > \tilde{h}(z)$.

[3]Alternatively, we may use Corollary 2.17 or 2.19.

To conclude just observe that u_* agrees on $B_r(x)$ with a harmonic function implying. Because $B_r(x) \subset\subset \Omega$ is arbitrary we see that u is harmonic in Ω. An analogous argument applies to u^*. □

On arbitrary domain Ω one can show that $u_* \neq u^*$, in particular, u_* does not agree with g on the boundary $\partial\Omega$.

Definition 2.22 (Regular point)**.** A point $x_0 \in \partial\Omega$ is called *regular* if for all $g \in C^0(\partial\Omega)$ and all $\epsilon > 0$ there are functions $u_\pm^\epsilon \in S_\pm$ such that $|g(x_0) - u_\pm^\epsilon(x_0)| \le \epsilon$.

Proposition 2.23. *The Dirichlet problem is solvable for all $g \in C^0(\partial\Omega)$ if and only if each point in $\partial\Omega$ is regular.*

Proof. The "only if" direction follows by taking the Dirichlet solution with boundary data g as $u_\pm^\epsilon$. For the opposite direction let $x_0 \in \partial\Omega$ and $u_-^{\frac{1}{n}} \in S_-$, $n \in \mathbb{N}$, as in the definition of regularity of $x_0 \in \partial\Omega$. Then $\max\{u_-^{\frac{1}{n}}, u_*\} \in S_-$ for all $n \in \mathbb{N}$. But the definition shows $u_-^{\frac{1}{n}} \le u_*$ showing that

$$0 \le g(x_0) - u_*(x_0)\big|_{\partial\Omega} \le g(x_0) - u_-^{\frac{1}{n}}(x_0) \le \frac{1}{n}$$

which shows $g(x_0) = u_*(x_0)$. Note that the same applies to $u_+^{\frac{1}{n}}$ and u^*.

To see that u_* is continuous each $x_0 \in \partial\Omega$ (and thus $u_* \in C^0(\bar\Omega)$) it suffices to show that for $x_n \in \Omega$ with $x_n \to x_0$ it holds $u_*(x_n) \to u_*(x_0) = g(x_0)$. Observe

$$u_-^\epsilon \le u_* \le u^* \le u_+^\epsilon$$

where $u_\pm^\epsilon \in S_\pm$ is as in definition of regularity of x_0. Because both u_-^ϵ and u_+^ϵ are continuous in $\bar\Omega$ there is a $\delta > 0$ such that

$$|u_\pm^\epsilon(x) - u_\pm^\epsilon(x_0)| \le \epsilon \qquad \text{for all } x \in B_\delta(x_0) \cap \bar\Omega.$$

Assume chose now $N > 0$ such that $x_n \in B_\delta(x_0)$ for $n \ge N$. Then we obtain

$$\begin{aligned} u_*(x_n) - u_*(x_0) &\le u_+^\epsilon(x_n) - g(x_0) \\ &\le |u_+^\epsilon(x_n) - u_+^\epsilon(x_0)| + |u_+^\epsilon(x_0) - g(x_0)| \\ &\le 2\epsilon \end{aligned}$$

and similarly

$$\begin{aligned} u_*(x_0) - u_*(x_n) &\le g(x_0) - u_-^\epsilon(x_n) \\ &\le |u_-^\epsilon(x_n) - u_-^\epsilon(x_0)| + |u_-^\epsilon(x_0) - g(x_0)| \\ &\le 2\epsilon. \end{aligned}$$

Thus $u_*(x_n) \to u_*(x_0)$. □

The proof shows that it suffices to only look at $u_-^\epsilon \in S_-$. More generally we want to show that it suffices to look at solution u_* obtained from Perron's Method with $g = -d(\cdot, x_0)\big|_{\partial\Omega}$.

Definition 2.24 (Barrier function)**.** A subharmonic function $b \in C^2(\Omega) \cap C^0(\bar\Omega)$ is called a lower barrier at $x_0 \in \partial\Omega$ if $b(x) < 0$ for all $x \in \partial\Omega\backslash\{x_0\}$ and $b(x_0) = 0$. A superharmonic function b is an upper barrier at $x_0 \in \partial\Omega$ if $-b$ is a lower barrier.

Proposition 2.25. *If there is a lower barrier function b at $x_0 \in \partial\Omega$ then x_0 is regular.*

Proof. By continuity there is a $\delta > 0$ such that $|g(x) - g(x_0)| \leq \epsilon$ whenever $x \in B_\delta(x_0)$. Furthermore, there is a $k > 0$ such that

$$2 \sup_{\partial\Omega} |g| + \sup_{\partial\Omega \setminus B_\delta(x_0)} kb \leq 0.$$

Such a k exists since $b < 0$ on $\partial\Omega\setminus B_\delta(x_0)$.

Define

$$u := g(x_0) - \epsilon + kb.$$

Then u is subharmonic. In order to show that $u \in S_-$ it suffices to show $u\big|_{\partial\Omega} \leq g$. Since $kb \leq 0$ we have for $x \in B_\delta(x_0) \cap \partial\Omega$

$$g(x) - u(x) = g(x) - g(x_0) + \epsilon - kb \geq 0$$

and

$$|u(x_0) - g(x_0)| = \epsilon.$$

If, however, $x \in \partial\Omega\setminus B_\delta(x_0)$ then

$$u(x) - g(x) = -\epsilon - g(x) + g(x_0) + kb \leq 2 \sup_{\partial\Omega} |g| + \sup_{\partial\Omega\setminus B_\delta(x_0)} kb \leq 0.$$

An analogous argument shows that

$$v = g(x_0) + \epsilon - kb \in S_+$$

and

$$|v(x_0) - g(x_0)| = \epsilon.$$

□

The proof of Proposition 2.23 shows that a (lower) barrier exists at $x_0 \in \partial\Omega$ if and only if $x_0 \in \partial\Omega$. Indeed, choose $g = -d(\cdot, x_0)\big|_{\partial\Omega} \in C^0(\partial\Omega)$ and apply the first step of Perron's Method. Then u_* is harmonic in $\partial\Omega$ and $u_* \leq g$ on $\partial\Omega$. In particular, $u_* < 0$ on $\partial\Omega\setminus\{x_0\}$. Thus if x_0 is regular then $u_*(x_0) = g(x_0) = 0$ so that u_* is a lower barrier at x_0.

In $\mathbb{R}^n$ there is a sufficient condition for a boundary point to admit a (lower) barrier.

Definition 2.26 (Exterior ball condition)**.** The domain Ω satisfies the *exterior ball condition* at $x_0 \in \partial\Omega$ if there is a $y \in \mathbb{R}^n\setminus\Omega$ and an $R > 0$ such that

$$\bar{B}_R(y) \cap \bar{\Omega} = \{x_0\}.$$

Lemma 2.27. *Assume Ω satisfies the exterior ball condition at $x_0 \in \partial\Omega$ then x_0 is regular.*

Proof. Observe that the function

$$u(x) = \begin{cases} \|x - y\|^{2-n} - R^{2-n} & n > 2 \\ \log \frac{\|x-y\|}{R} & n = 2 \end{cases}$$

then u is in $C^\infty(\bar{\Omega})$ and $\Delta u = 0$ in Ω. Furthermore, $u < 0$ outside of $\bar{B}_R(y)$ and $u(x_0) = 0$ showing that u is a barrier at x_0 and thus x_0 regular. □

Poincaré's Method. In Perron's Method we replaced locally v_n by a harmonic function and then used this to conclude that locally the limit is harmonic and agrees with u_*. Instead of using local argument and define u_* via the (pointwise) maximum of subharmonic functions, we could think of replacing successively an initial function by a function that is locally harmonic. If we do it everywhere sufficiently often this should give a non-decreasing sequence of uniformly bounded subharmonic functions. The limit should then be harmonic. This method is called Poincaré's Method.

For this let $B_n := B_{r_n}(x_n) \subset\subset \Omega$ be balls such that $\Omega = \cup_{n\in\mathbb{N}} B_n$. Define now a sequence

$$n_k = 1, 21, 2, 3, 1, 2, 3, 4, \ldots, 1, \ldots, n, 1, \ldots$$

Let $g \in C^0(\bar{\Omega})$ and set $u_0 = g$. For $k \geq 1$ let h_k be the harmonic function on B_{n_k} with boundary data $u_{k-1}\big|_{\partial\Omega}$ and define

$$u_k = \begin{cases} u_{k-1} & x \notin \bar{\Omega}\backslash B_{n_k} \\ h_k & x \in B_{n_k}. \end{cases}$$

Proposition 2.28 (Poincaré's Method)**.** *Assume each point in $\partial\Omega$ is regular then the sequence $(u_k)_{k\in\mathbb{N}}$ converges to a function u which is harmonic in Ω and satisfies $u\big|_{\partial\Omega} = g\big|_{\partial\Omega}$.*

Proof. The maximum principle on B_{n_k} implies that

$$g = u_0 \leq u_1 \leq \cdots \leq u_k \leq \cdots \leq \sup_{\Omega} g.$$

Define

$$u(x) = \sup_{k\in\mathbb{N}} u_k(x) = \lim_{k\to\infty} u_k(x)$$

Assume first $g \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is subharmonic in Ω. Then by the Replacement Lemma (Lemma 2.20) u_k is mean-value subharmonic. Furthermore, for each $x \in \Omega$ there is a sequence $(k_l)_{n\in\mathbb{N}}$ such that u_{k_l} is harmonic on $B = B_{n_{k_l}}$ with $x \in B$. Thus by Harnack's Convergence Theorem the sequence $(u_{k_l})_{l\in\mathbb{N}}$ converges uniformly on B to a harmonic function h. However, this implies

$$u(x) = \lim_{k\to\infty} u_k(x) = \lim_{l\to\infty} u_{k_l}(x) = h(x)$$

and thus u is harmonic on B. Because $x \in \Omega$ is arbitrary we see that u is harmonic. The existence of barriers at $x_0 \in \partial\Omega$ then implies as above that $u(x_0) = g(x_0)$.

Now suppose $g \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\Delta g \geq -\lambda$. Then $g_0 = g + \frac{\lambda}{2n}\|\cdot\|^2$ satisfies $\Delta g_0 = \Delta g + \lambda \geq 0$, i.e. g_0 is subharmonic. As $\Delta\|\cdot\|^2 \geq 0$ we can solve first via Poincaré's Method with g_0 and obtain a harmonic function u_0 with boundary data $g_0\big|_{\partial\Omega}$. Then solve via Poincaré's Method with $\frac{\lambda}{2n}\|\cdot\|^2$ to obtain a harmonic function $\breve{u}$ with boundary data $\frac{\lambda}{2n}\|\cdot\|^2\big|_{\partial\Omega}$.

By linearity we see that $u_0 - \breve{u}$ is harmonic with boundary data $g\big|_{\partial\Omega}$.

Finally if $g \in C^0(\bar{\Omega})$ then there exists a sequence of functions $g_n \in C^\infty(\bar{\Omega})$ with $g_n \leq g_{n+1}$, $n \in \mathbb{N}$ such that $g = \lim_{n\to\infty} g_n$. Via Poincaré's Method we obtain a sequence of harmonic functions u_n with boundary data $g_n\big|_{\partial\Omega}$ converging monotonically to a harmonic function u with boundary data $g\big|_{\partial\Omega}$. □

Remark. If $\mathcal{D}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$ denotes the Dirichlet energy of u then Poincaré's Method satisfies

$$\mathcal{D}(g) \geq \mathcal{D}(u_1) \geq \cdots \geq \mathcal{D}(u_k) \geq \mathcal{D}(u)$$

provided that u_k has a well-defined Dirichlet energy.

Generalization of Perron's Method to solution operators. Assume $\Omega \subset \mathbb{R}^n$ is an open set

Solution operator:

Given any ball $B_r(x) \subset\subset \Omega$ there is a *linear* operator $P_B : C^0(\partial B_r(x)) \to C^0(\bar{B}_r(x))$ such that given $g \in C^0(\partial B_r(x))$ and $h := P_{B_r(x)} g$ it holds

- if $g \equiv c$ for some $c \in \mathbb{R}$ then
$$P_{B_r(x)}(g) \equiv c \quad \text{ on } \bar{B}_r(x).$$
- if $g \geq 0$ then $h \geq 0$ and h satisfies the *Harnack inequality* on $\Omega^+ = \operatorname{int}\{h \geq 0\}$, i.e. for $\Omega' \subset\subset \Omega^+$ there is a constant $C > 0$, not depending on h, such that
$$\sup_{\Omega'} h \leq C \inf_{\Omega'} h.$$

Weak subharmonicity:

A function $u \in C^0(\Omega)$ is called *weakly subharmonic* if for all $B_r(x) \subset\subset \Omega$ it holds

$$u \leq P_B g \quad \text{ on } \bar{B}_r(x)$$

whenever $u\big|_{\partial B_r(x)} \leq g$. If both u and $-u$ are weakly subharmonic then u is called *weakly harmonic.*

Using those ingredients it is possible to show the following generalized variant of Perron's Method.

Theorem (Perron's Method)**.** *Let $g \in C^0(\partial\Omega)$ and define*

$$S_- = \{u \in C^0(\bar{\Omega}) \,|\, u \text{ is weakly subharmonic and } u\big|_{\partial\Omega} \leq g\}.$$

Then the function u_* defined by

$$u_*(x) = \sup_{u \in S_-} u(x)$$

is weakly harmonic in Ω.

3. Classical Maximum Principles

3.1. **Elliptic maximum principles.** Let L be an elliptic operator on function in $C^2(\Omega)$ such that

$$\lambda(x) \sum_{i=1}^n \xi_i \xi_i \leq \sum a^{ij}(x) \xi_i \xi_j \leq \Lambda(x) \sum_{i=1}^n \xi_i \xi_i$$

and

$$\boldsymbol{b} := \sup_{k=1}^n \sup_{x \in \Omega} \frac{|b^k(x)|}{\lambda(x)}.$$

Lemma 3.1. *Assume $c = 0$. If $u \in C^2(\Omega)$ with $Lu > 0$ in Ω then u does not assume a maximum in Ω, i.e. for all $x \in \Omega$ it holds $u(x) < \sup_\Omega u$. In particular, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Lu > 0$ satisfy the strong maximum principle*

$$u(x) < \sup_\Omega u = \sup_{\partial\Omega} u.$$

Proof. Assume $u(x) = \sup_\Omega u$ for some $x \in \Omega$. Then the Hessian $D^2u(x)$ is non-positive. Furthermore, $\partial_i u(x) = 0$ for all $i = 1, \ldots, n$. Hence

$$Lu(x) = \sum a^{ij}(x)\partial_{ij}u(x) = \mathsf{tr}(A(x) \cdot D^2u(x)) \leq 0$$

since $A(x) = (a^{ij}(x))_{i,j=1}^n$ is symmetric positive definite. □

Lemma 3.2. *The function*

$$v : x = (x_1, \ldots, x_n) \mapsto e^{-\gamma x_1}$$

satisfies $Lv > 0$ *provided* $\gamma \boldsymbol{b} < 1$ *and* $c = 0$.

Proof. Just note that

$$\begin{aligned} Lv(x) &= \gamma^2 a^{11}(x) + \gamma b^1(x) \\ &\geq \gamma\lambda(x)\left(1 - \gamma\frac{|b^1(x)|}{\lambda(x)}\right) > 0. \end{aligned}$$

□

Theorem 3.3 (Weak Maximum Principle)**.** *Assume* $c = 0$ *and* $\boldsymbol{b} < \infty$. *If* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies* $Lu \geq 0$ *then* u *satisfies the weak maximum principle, i.e.*

$$\sup_\Omega u = \sup_{\partial\Omega} u.$$

Proof. Choose $\gamma \in (0, \frac{1}{\boldsymbol{b}})$ and let v be as in the previous lemma. Then

$$L(u + \epsilon v) > 0$$

so that

$$\sup_\Omega(u + \epsilon v) = \sup_{\partial\Omega}(u + \epsilon v).$$

Letting $\epsilon \to 0$ implies the result. □

Corollary 3.4 (Uniqueness of the Dirichlet problem)**.** *If* $c = 0$ *and* $\boldsymbol{b} < \infty$ *then* $Lu = Lv$ *and* $u\big|_{\partial\Omega} = v\big|_{\partial\Omega}$ *for functions* $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *implies* $u = v$ *on* Ω.

Proof. Observe that

$$L\left[\pm(u - v)\right] = 0$$

so that the maximum principle implies

$$\begin{aligned} \sup_\Omega |u - v| &= \max\left\{\sup_\Omega\left[+(u - v)\right], \sup_\Omega\left[-(u - v)\right]\right\} \\ &= \max\left\{\sup_{\partial\Omega}\left[+(u - v)\right], \sup_{\partial\Omega}\left[-(u - v)\right]\right\} = 0 \end{aligned}$$

implying the result. □

Definition 3.5 (Interior Ball Condition)**.** A point $x_0 \in \partial\Omega$ satisfies the *interior ball condition* if there is a $B_R(y) \subset \Omega$ such that $\bar{B}_R(y) \cap \partial\Omega = \{x_0\}$.

We make the first observation.

Lemma 3.6. *If* $x_0 \in \partial\Omega$ *satisfies the interior ball condition and the (outer) unit normal* ν *of* $\partial\Omega$ *at* x_0 *exists then* $\nu = \frac{x_0 - y}{R}$ *where* $B_R(y) \subset \Omega$ *and* $\bar{B}_R(y) \cap \partial\Omega$. *Furthermore,* ν *is also the unit normal of* $\partial B_R(y)$ *at* x_0.

Remark. A point $x_0 \in \partial\Omega$ satisfies the interior ball condition if $x_0 \in \partial(\mathbb{R}^n \backslash \bar{\Omega})$ saisfies the exterior ball condition. Since the unit normal ν of $\partial\Omega$ at x_0 exists if and only if the unit normal $\bar{\nu}$ of $\partial(M\backslash\bar{\Omega})$ at x_0 exists the exterior ball condition also gives

Lemma 3.7. *Assume $c = 0$ and $\boldsymbol{b} < \infty$. For fixed $0 < \rho < R$ there is a sufficiently large $\alpha \gg 1$ such that the function*

$$v(x) = e^{-\alpha\|x-y\|^2} - e^{-\alpha R^2}$$

satisfies $Lu \geq 0$ on $B_R(y)\backslash B_\rho(y)$ and $\partial_\nu v(x_0) = -2\alpha R e^{-\alpha R^2} < 0$ for all $x_0 \in \partial B_R(y)$.

Proof. For $x \in B_R(y)\backslash B_\rho(y)$ it holds

$$\begin{aligned}
Lv(x) &= e^{-\alpha\|x-y\|^2}\left[4\alpha^2 \sum \left(a^{ij}(x)(x_i-y_i)(x_j-y_j)\right) - 2\alpha \sum_{i=1}^n \left(a^{ii}(x) + b^i(x_i-y_i)\right)\right] \\
&\geq e^{-\alpha\|x-y\|^2}\left[4\alpha^2 \sum \left(a^{ij}(x)(x_i-y_i)(x_j-y_j)\right) - 2\alpha \sum_{i=1}^n \left(a^{ii}(x) + \sup_{i=1}^n |b^k(x)|\|x_i-y_i\|\right)\right] \\
&\geq e^{-\alpha\|x-y\|^2}\left[4\alpha^2\lambda(x)\|x-y\|^2 - 2\alpha \sum_{i=1}^n \left(a^{ii}(x) + \sup_{i=1}^n |b^k(x)|\|x_i-y_i\|\right)\right] \\
&\geq e^{-\alpha\|x-y\|^2}\left[4\alpha^2\lambda(x)\rho^2 - 2\alpha \sum_{i=1}^n \left(a^{ii}(x) + \sup_{i=1}^n |b^k(x)|R\right)\right] \\
&\geq e^{-\alpha\|x-y\|^2} 2\alpha\lambda(x)\left[2\alpha\rho^2 - \sum_{i=1}^n \left(\frac{a^{ii}(x)}{\lambda(x)} + \boldsymbol{b}R\right)\right]
\end{aligned}$$

Since

$$a^{ii}(x) \geq \lambda(x)$$

we may choose $\alpha \gg 1$ so that

$$2\alpha\rho^2 - \sum_{i=1}^n \left(\frac{a^{ii}(x)}{\lambda(x)} + \boldsymbol{b}R\right) \geq 0$$

implying

$$Lv \geq 0.$$

□

Lemma 3.8. *Assume $c = 0$ and $\boldsymbol{b} < \infty$. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy $Lu \geq 0$. In addition, assume the following holds for $x_0 \in \partial\Omega$*

- *the point $x_0 \in \partial\Omega$ satisfies the interior ball condition*
- *$u(x_0) > u(x)$ for all $x \in \Omega$.*

Then $\partial_\nu u(x_0) > 0$ provided the unit normal at $x_0 \in \partial\Omega$ exists.

Proof. Let $B_R(y)$ be given by the interior ball condition of x_0 and pick $\rho > 0$ and choose α such that v as in the previous lemma satisfies $Lv \geq 0$.

Set $A := B_R(y)\backslash B_\rho(y)$ and observe that

$$u - u(x_0) < 0 \quad \text{on } \partial B_\rho(y).$$

Furthermore, $v = 0$ on $\partial B_R(y)$. Thus there is an $\epsilon > 0$ such that

$$u - u(x_0) + \epsilon v \le 0 \quad \text{on } \partial A.$$

Since $L(u - u(x_0) + \epsilon v) \ge 0$ the weak maximum principle implies

$$u - u(x_0) + \epsilon v \le 0 \quad \text{on } A.$$

Assuming the unit normal ν at $x_0 \in \partial\Omega$ exists this yield

$$\partial_\nu u \ge -\epsilon \partial_v v > 0.$$

□

Theorem 3.9 (Strong Maximum Principle). *Assume L is an elliptic operator with $c = 0$ and $\boldsymbol{b} < \infty$. If $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ satisfies $Lu \ge 0$ and $u(x^*) = \sup_\Omega u$ for some $x^* \in \Omega$ then u is constant in Ω.*

Proof. It suffices to prove that

$$\Omega^+ = \{x \in \Omega \,|\, u(x) < u(x^*)\}$$

is empty.

Assume by contradiction Ω^+ is non-empty. Note by continuity Ω^+ is open.

We first claim there is a $y \in \Omega^+$ such that

$$d(y, \partial\Omega^+) < d(y, \partial\Omega).$$

Indeed, by assumption $\partial\Omega^+ \backslash \partial\Omega \neq \varnothing$. Taking a point $x \in \partial\Omega^+ \backslash \partial\Omega \subset \Omega$ and using compactness of $\partial\Omega$ we may take $y \in B_\epsilon(x_0) \cap \Omega^+$ for $\epsilon < \frac{1}{2} d(x, \partial\Omega)$.

Let $x_0 \in \partial\Omega^+$be such that $R := d(y, \partial\Omega^+) = d(y, x_0)$. Then $u(x_0) = u(x^*) = \sup_\Omega u$ and u restricted to $\tilde\Omega = B_R(y)$ and $x_0 \in \partial\tilde\Omega$ satisfies the assumptions of the previous lemma. Since the unit normal at $x_0 \in \partial\tilde\Omega = \partial B_R(y)$ exists it holds

$$\partial_\nu u(x_0) > 0.$$

However, x_0 is an interior point of Ω so that $\partial_i u(x_0) = 0$. This is a contradiction and implies that Ω^+ is empty and thus u must be constant. □

We complete this subsection with the case $c \le 0$.

Theorem 3.10 (Maximum principle for $c \ge 0$). *Assume Ω is open, bounded and connected, and L is an elliptic operator with $c \le 0$ and $\boldsymbol{b} < \infty$. Then for all $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ with $Lu \ge 0$ it holds*

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+$$

where $u^+ = \max\{0, u\}$. Furthermore, if $u(x_0) = \sup_\Omega u > 0$ for some $x_0 \in \Omega$ then u is constant in Ω.

Proof. Let

$$\Omega^+ = \{u > 0\}.$$

If $\Omega^+ = \varnothing$ then $u \le 0$ so that

$$\sup_\Omega u \le 0 = \max\{0, u^+\} = \sup_{\partial\Omega} u^+.$$

If $\Omega^+ \neq \varnothing$ then

$$\tilde L u := Lu - cu \ge 0 \quad \text{on } \Omega^+$$

and $\tilde{L} = L - c$ is an elliptic operator with $c = 0$. Thus

$$0 < \sup_{\Omega^+} u = \sup_{\partial\Omega^+} u.$$

Since Ω is bounded, we see that $\partial\Omega^+$ is compact so that there is an $x \in \partial\Omega^+$ with $u(x_0) = \sup_{\Omega^+} u > 0$.

To see that $x_0 \in \partial\Omega$ observe that every point $y \in \partial(\Omega\backslash\Omega^+)$ satisfies $u(y) \leq 0$. Thus

$$x \in \partial\Omega \cap \partial\Omega^+$$

and

$$\sup_{\partial\Omega} u = \sup_{\partial\Omega\cap\partial\Omega^+} u = u(x)$$

which implies $\sup_\Omega u = \sup_{\partial\Omega} u^+$ as $u^+ = u$ on $\partial\Omega \cap \partial\Omega^+$.

If $u(x_0) = \sup_\Omega u > 0$ for some $x_0 \in \Omega$ then $x_0 \in \Omega^+$ and u is constant on the connected component $\Omega^+_{x_0}$ containing x_0. Thus

$$\Omega^+_{x_0} = u^{-1}((-\infty, 0)) = u^{-1}(\{u(x_0)\})$$

is both closed and open in Ω. Hence by connectedness $\Omega = \Omega^+_{x_0}$, i.e. u is constant on Ω. □

Corollary 3.11. *Assume $c \leq 0$. If $v, u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Lu = Lv$ and $u\big|_{\partial\Omega} = v\big|_{\partial\Omega}$ then $u = v$ on Ω.*

Proof. Since $L(u - v) = 0$ we see that

$$\sup_\Omega (u - v)_+ \leq \sup_{\partial\Omega} (u - v)^+ = 0$$

and

$$\sup_\Omega (v - u)_+ \leq \sup_{\partial\Omega} (v - u)^+ = 0$$

which shows $\sup_\Omega |u - v| = 0$ and thus $u = v$. □

3.2. **Parabolic maximum principles.** In the following we assume L is an elliptic operator so that $\partial_t - L$ is a parabolic operator on $Q = (0, T) \times \Omega$.

Lemma 3.12. *Assume $c = 0$. If $u \in C^2(Q)$ with $\partial_t - Lu < 0$ in Q then u does not assume a maximum in Q, i.e. for all $x \in \Omega$ it holds $u(x) < \sup_\Omega u$. In particular, $u \in C^2(Q) \cap C^0(\bar{Q})$ with $Lu > 0$ satisfy the strong maximum principle*

$$u(t, x) < \sup_Q u = \sup_{\partial' Q} u \quad \textit{for all } (t, x) \in Q$$

where $\partial' Q = (0, T) \times \partial\Omega \cup \{0\} \times \Omega$ is the parabolic boundary of Q.

Proof. Define $Q' = (0, T') \times \Omega$ for $T' \in (0, T)$. Assume $u(t_0, x_0) = \sup_Q u$ for some $(t_0, x_0) \in Q$. Then

$$\partial_t u, \partial_i u = 0$$

and

$$0 > \partial_t u(t, x) - Lu(t, x) = -\mathsf{tr}(A(t, x) \cdot D_x^2 u(t, x)) \geq 0$$

which is a contradiction. Here $D_x^2 u(t, x)$ denotes the Hessian matrix $(\partial_{ij} u(t, x))_{i,j=1}^n$ in the space variables only.

Thus

$$\sup_{Q'} u = \sup_{\partial Q'} u.$$

By the same argument $u(T', x) < \sup_Q u$ so that

$$\sup_{Q'} u = \sup_{\partial' Q'} u.$$

The claim follows by letting T' tend to T. □

Theorem 3.13 (Weak Maximum Principle)**.** *Assume $c = 0$ and $\boldsymbol{b} < \infty$. If $u \in C^2(Q) \cap C^0(\bar{Q})$ satisfies $Lu \geq 0$ then u satisfies the weak maximum principle, i.e.*

$$\sup_{\Omega} u = \sup_{\partial' Q} u.$$

Proof. Note that for $v(t, x) = -t$ and $\epsilon > 0$ it holds

$$\partial_t(u + \epsilon v) - L(u + \epsilon v) < 0.$$

Thus

$$\begin{aligned} \sup_Q u &= \lim_{\epsilon \to 0} \sup_Q (u + \epsilon v) \\ &= \lim_{\epsilon \to 0} \sup_{\partial' Q} (u + \epsilon v) = \sup_{\partial' Q} u. \end{aligned}$$

□

By the same argument we also obtain the weak maximum principle if $c \leq 0$.

Theorem 3.14. *Assume $c \leq 0$ and $\boldsymbol{b} < \infty$. If $u \in C^2(Q) \cap C^0(\bar{Q})$ satisfies $Lu \geq 0$ then u satisfies the weak maximum principle, i.e.*

$$\sup_{\Omega} u \leq \sup_{\partial' Q} u^+.$$

Corollary 3.15 (Uniqueness)**.** *Assume $c \leq 0$ and $\boldsymbol{b} < \infty$. If $\partial_t u - Lu = \partial_t v - Lv$ for function $u, v \in C^2(Q) \cap C^0(\bar{Q})$ with $u\big|_{\partial' Q} = v\big|_{\partial' Q}$ then $u = v$ on Q.*

For the strong maximum principle we look at the following function: Given $(s, y) \in Q$ and $R > 0$ define

$$v(t, x) := e^{-\alpha r(t,x)} - e^{-\alpha R^2}$$

where in the set

$$Q_{y,s,R} = \left\{ (t, x) \in Q \,|\, \|x - y\| > \frac{R}{2}, r(t, x)^2 < R^2, t < s \right\}$$

where $r(t, x) = \left(\|x - y\|^2 + \eta^2(s - t)\right)^{\frac{1}{2}}$ with η^2 to be determined later on.

As in the elliptic case it is possible to force v to be a parabolic subsolution.

Lemma 3.16. *Assume L is uniformly elliptic, $c = 0$ and $\boldsymbol{b} < \infty$ (thus $\sup_{k=1}^n |b^k| < \infty$. Then it holds*

$$(\partial_t - L)v < 0$$

for sufficiently large α.

Proof. It holds

$$\begin{aligned}(\partial_t - L)v &= e^{-\alpha r^2}\left[-4\alpha^2 \sum \left(a^{ij}(x)(x_i-y_i)(x_j-y_j)\right) + 2\alpha\sum_{i=1}^n \left(a^{ii}(x)+b^i(x_i-y_i)+1\right)\right]\\ &\le 2\alpha e^{-\alpha r^2}\left[-2\alpha\lambda\|x-y\|^2 + \sum_{i=1}^n\left(a^{ii}(x)+\sup_{i=1}^n|b^k(x)|\|x_i-y_i\|+1\right)\right]\\ &\le 2\alpha e^{-\alpha r^2}\left[-2\alpha\lambda\frac{R^2}{2} + \sum_{i=1}^n\left(a^{ii}(x)+\sup_{i=1}^n|b^k(x)|R+1\right)\right]\end{aligned}$$

which is negative for α sufficiently large. □

Note that the set $\bar{Q}_{y,s,R}$ is a cone with tip at $(s-\frac{R^2}{\eta^2}, y)$ and base $\{s\}\times\bar{B}_R(y)$. Furhtermore, the construction shows

$$\partial_v v(x,s) < 0$$

for $v = \frac{x-y}{\|x-y\|}$ where ∂_v denotes the derivative in direction v. We want to this to show that at boundary $\{s\}\times\partial\bar{B}_R(y)$ of the base and maximum point must have non-vanishing derivative.

Lemma 3.17. *Assume L is uniformly elliptic, $c = 0$ and $\boldsymbol{b} < \infty$. Let $u \in C^2(\Omega)$ satisfy $\partial_t - Lu \le 0$ and assume the for $(t_0, x_0) \in \Omega\times(0,T)$ there is a $(y, t_0) \in Q$ and $R > 0$ such that for $\|x_0 - y\| = R$ it holds*

$$\begin{aligned}&Q_{y,t_0,R} \subset Q\\ &(t_0, x_0) \in \partial Q_{y,t_0,R}\end{aligned}$$

and $u(t_0, x_0) > u(t, x)$ for all $(t, x) \in Q_{x_0,t_0,R}$. Then $\partial_{\tilde{\nu}}u(t, x_0) > 0$ where $\tilde{\nu} = (0, \frac{x_0-y}{\|x-y\|})$.

Proof. Observe that

$$\begin{aligned}\partial Q_{y,t_0,R} &= \{(t,x)\in Q \,|\, r(t,x) = R, t\le t_0\} \cup \{(t,x)\in\bar{Q} \,|\, \|x-y\| = \frac{R}{2}, t\le t_0\}.\\ &= S_1\cup S_2\end{aligned}$$

and

$$\begin{aligned}&v = 0 \qquad \text{on } S_1\\ &v \le e^{-\alpha\rho^2} - e^{-\alpha R^2} \qquad \text{on } S_2.\end{aligned}$$

Since

$$\begin{aligned}u - u(t_0,x_0) &\le 0 \qquad \text{on } S_1\\ u - u(t_0,x_0) &< 0 \qquad \text{on } S_2\end{aligned}$$

and S_2 is compact

$$\begin{aligned}(\partial_t - L)u - u(t_0,x_0) + \epsilon v &\le 0 \qquad \text{in } Q_{y,t_0,R}\\ u - u(t_0,x_0) + \epsilon v &\le 0 \qquad \text{on } \partial' Q_{y,t_0,R}.\end{aligned}$$

for sufficiently small $\epsilon > 0$.

Then the weak maximum principle implies $u - u(t_0, x_0) + \epsilon v \le 0$ in $Q_{y,t_0,R}$ so that

$$\partial_{\tilde{\nu}}u \ge -\epsilon\partial_{\tilde{\nu}}v > 0.$$

□

Theorem 3.18 (Strong Parabolic Maximum Principle)**.** *Assume L is uniformly elliptic with $c = 0$ and $\boldsymbol{b} < \infty$. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $(\partial_t - L)u \leq 0$ and $u(t^*, x^*) = \sup_Q u$ for some $(t^*, x^*) \in Q$ then u is constant in Q.*

Proof. Define an open subset of Q by

$$Q^+ = \{(t,x) \in Q \,|\, u(t,x) < u(t^*, x^*)\}.$$

We need to show that Q^+ is empty. Assume this is not the case then there is an $(y,s) \in Q^+$ and $\epsilon > 0$ such that $(y, s') \in Q^+$ for all $s' \in [s - \epsilon, s]$.

By compactness of $\bar{Q}$ there is a maximal $\eta > 0$ such that for $R = \sqrt{\epsilon} \cdot \eta$

$$Q_{y,s,R} \subset Q^+.$$

As in the elliptic case we can find $(y,s) \in Q^+$ such that

$$\partial Q_{y,s,\frac{R}{2},R} \cap \partial Q = \varnothing.$$

In particular, there is a point $(t_0, x_0) \in \partial Q_{y,s,\frac{R}{2},R}$ with $u(t_0, x_0) = u(t^*, x^*)$. Note that the choice of $R = \sqrt{\epsilon} \cdot \eta$ ensures $t_0 \neq s - \epsilon$ and $x_0 \neq y$.

Then for $R' = \|y - x_0\| > 0$ it holds

$$Q_{y,t_0,\frac{R'}{2},R'} \subset Q_{y,s,\frac{R}{2},R} \subset Q^+$$

and $(t_0, x_0) \in \partial Q_{y,t_0,\frac{R'}{2},R'}$. In this case the previous lemma yields

$$\partial_{\tilde{\nu}} u(t_0, x_0) > 0.$$

However, $(t_0, x_0) \in Q$ is a maximum point of u so that

$$\partial_{\tilde{\nu}} u(t_0, x_0) = 0$$

which is a contradiction. □

4. Sobolev theory in $\mathbb{R}^n$

4.1. **Banach spaces.**

Definition 4.1 (Banach space)**.** A complete normed space $(X, \|\cdot\|)$ is called a *Banach space*, i.e. X is a vector space and $\|\cdot\|$ is a norm such that the induced metric d defined by $d_{\|\cdot\|}(v,w) = \|v - w\|$ makes $(X, d_{\|\cdot\|})$ into a complete metric space.

Remark (Construction of Banach spaces). Using completion we can obtain from a general normed space $(X, \|\cdot\|)$ a (unique up to linear isomorphism) Banach space $(\tilde{X}, \|\cdot\|')$ such that X seen as a subset of $\tilde{X}$ is dense in X and the norms $\|\cdot\|$ and $\|\cdot\|'$ agree on X.

4.2. **Function spaces.** Let $A \subset \mathbb{R}^n$, e.g. $A = \Omega$ or $A = \bar{\Omega}$, and define the following spaces:

- $C^0(A) = \{$space of continuous functions on $A\}$.
- $C^k(A) = \{u \in C^0(A) \,|\, \partial_I u \in C^0(A)$ for all multi indices I with $|I| \leq k\}$.
- for $\alpha \in (0,1]$: $C^{0,\alpha}(A) = \{u \in C^0(A) \,|\, \sup_{x,y \in A} \frac{|u(x) - u(y)|}{\|x-y\|^\alpha} < \infty\}$. If $\alpha \in (0,1)$ the space $C^{0,\alpha}(A) = C^\alpha(A)$ is called the space of Hölder function and $C^{0,1}(A)$ is called the space of Lipschitz functions.
- $C^{k,\alpha}(A) = \{u \in C^k(A) \,|\, \partial_I u \in C^{0,\alpha}(A)$ for all multi indices I with $|I| = k\}$.

- $L^0(A) = L^0(A, \lambda^n\big|_A) = \{$space of equivalence classes of measurable functions on $A\}$ (two function $u, v \in L^0(A)$ are equivalent if $\lambda^n\big|_A(\{u \neq v\}) = 0$, i.e. $u = v$ almost everywhere.
- for $p \in (0, \infty)$: $L^p(A) = \{u \in L^0(A) \mid \int_A |u|^p d\lambda^n < \infty\}$
- $L^\infty(A) = \{u \in L^0(A) \mid \operatorname{ess\,sup} |u| < \infty\}$.

On each of those spaces there is a natural norm making the subspace of functions with finite norm into a Banach space[4].

- $\|u\|_{C^0} = \sup_{x\in A} |u(x)|$.
- $\|u\|_{C^k} = \sup_{x\in A, |I|\le k} |\partial_I u(x)|$.
- $\|u\|_{C^{0,\alpha}} = \max\{\sup_{x\in A} |u(x)|, \sup_{x,y\in A} \frac{|u(x)-u(y)|}{\|x-y\|^\alpha}$.
- $\|u\|_p = \left(\int_A |u|^p d\lambda^n\right)^{\frac{1}{p}}$, $p \in [1, \infty)$.
- $\|u\|_\infty = \operatorname{ess\,sup} |u|$.

4.3. **Properties of L^p-spaces.** The classical L^p-spaces are the space $L^p(A) = L^p(A, \lambda^n)$ for a Lebesgue measurable set $A \subset \mathbb{R}^n$ and λ^n the Lebesgue measure. Given two L^p-spaces we can define the L^p-product of two L^p-spaces $L^p(A)$ and $L^p(B)$ (possibly $A = B$) as the product vector space $L^p(A) \times L^p(B)$ equipped with the following norm

$$\|(f, g)\|_p = \left(\|f\|^p_{L^p(A)} + \|g\|^p_{L^p(B)}\right)^{\frac{1}{p}}.$$

It is not difficult to see that $(L^p(A) \otimes L^p(B), \|(\cdot,\cdot)\|)$ is a Banach space and very similar to the regular L^p-spaces. Furthermore, an iterated construction also shows that

$$\bigotimes_{i=1}^n L^p(A_i)$$

with corresponding norm is a Banach space.

Definition 4.2 (r-uniform convexity)**.** A Banach space $(X, \|\cdot\|)$ is said to be r-uniformly convex if there is a $C_p > 0$ such that for all $v, w \in X$ it holds

$$\left\|\frac{v+w}{2}\right\|^p + C_p\|v-w\|^p \le \frac{1}{2}\|v\|^p + \frac{1}{2}\|w\|^p.$$

Lemma 4.3. *For every $p \in (1, \infty)$ the L^p-spaces are r-uniformly convex for $r = \max\{2, p\}$ and*

$$C_p = \begin{cases} \frac{p-1}{4} & p \le 2 \\ \frac{1}{4} & p > 2. \end{cases}$$

Proof. The statement for $p \ge 2$ appeared in the exercises. An accessible proof can be found in *Sharp uniform convexity and smoothness inequalities for trace norms* by Ball–Carlen–Lieb in Inv.Math. (1994). □

Proposition 4.4. *Let C be a bounded, closed and convex subset in an r-uniformly convex Banach space. Then*

[4]If A is not compact then $C^k(A)$, $k \in \mathbb{N}$, might contain unbounded functions. The set of bounded continuous functions is usually denoted by $C^0_b(A)$. Furthermore, for $C^{0,\alpha}(A)$ one also needs to either add the C^0-norm (resp. take the maximum of the C^0-norm and the $C^{0,\alpha}$-seminorm).

Corollary 4.5. *Let $(C_i)_{i\in I}$ be a net of bounded, closed and convex subsets in an r-uniformly convex Banach space X such that $C_i \subset C_j$ whenever $j \geq i$. Then*

$$\bigcap_{i\in I} C_i \neq \varnothing.$$

Proof. We only prove the result for $I = \mathbb{N}$. We define a map $r : X \to [0,\infty)$ by

$$r(x) = \sup_{n\in\mathbb{N}} \inf_{z\in C_n} \|x - z\|.$$

This implies that there is a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in C_n$ and

$$|r(x) - r_n| \leq \frac{1}{n}$$

and

$$\|x - z_n\| \leq r_n + \frac{1}{n}$$

where $r_n = \inf_{z\in C_n} \|x - z_n\|$.

Since $z_m \in C_n$ whenever $m \geq n$ we see that $r(x)$ is bounded. We claim that (z_n) is a Cauchy sequence. Indeed, since X is complete, C_n is closed and $\{z_m\}_{m\geq n} \subset C_n$ we see that $z_n \to z$ for some $z \in X$ and $z \in C_n$ for all $n \in \mathbb{N}$. Thus $z \in \bigcap_{n\in\mathbb{N}} C_n$ which yields the result.

It remains to show that $(z_n)_{n\in\mathbb{N}}$ is Cauchy. Observe first that

$$r_n \leq \|x - \frac{z_n + z_m}{2}\|$$

and that $(r_n)_{n\in\mathbb{N}}$ (and hence $(r_n^r)_{n\in\mathbb{N}}$) is Cauchy. Fix $\epsilon > 0$ and let $n, m \geq \frac{1}{\epsilon}$ such that $r_m^r - r_n^r \leq \epsilon$. Then uniform convexity implies

$$\begin{aligned} C_r\|z_n - z_m\|^r &\leq \frac{1}{2}\|x - z_n\|^r + \frac{1}{2}\|x - z_m\|^r - \|x - \frac{z_n + z_m}{2}\|^r \\ &\leq \frac{1}{2}(r_m^r - r_n^r) + \frac{2}{n} \leq 3\epsilon. \end{aligned}$$

Thus we see that $(z_n)_{n\in\mathbb{N}}$ is Cauchy. □

Corollary 4.6. *For every closed convex subset C of an r-uniformly convex Banach space $(X, \|\cdot\|)$ and all $u \in X$ there is a unique $u_C \in C$ such that*

$$\|u - u_C\| = \inf_{v\in C} \|u - v\|.$$

Proof. Let $r_C(x) = \inf_{v\in C} \|u - v\|$ then

$$C_n = \bar{B}_{r_C(x)+\frac{1}{n}}(x) \cap C$$

satisfies the assumption of the previous statement. In particular, $\tilde{C} = \cap_{n\in\mathbb{N}} C_n \neq \varnothing$ which implies that there is a $u_C \in C$ with the required properties. To see that u_C is unique just observe that for any other $v \in \tilde{C}$ it holds $\frac{v+u_C}{2} \in \tilde{C}$. However, uniform r-convexity (even weak, the strict convexity of the norm) implies that

$$C_r\|v - u_C\|^r \leq 0$$

which means $v = u_C$. □

Remark. The proof of the previous two result are “equivalent”, i.e. it is possible to prove the latter without the former and then give a proof of the former using the statement of the latter.

4.4. **Sobolev spaces and minimizers of quadratic functionals.** Using Green's formula it is possible to show that a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a harmonic function in Ω with $u\big|_{\partial\Omega} = g$ if and only if it is the minimizer

$$v \mapsto \int_\Omega |\nabla v|^2 dx$$

among all function $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying $v\big|_{\partial\Omega} = u\big|_{\partial\Omega} = g$. Since $v - u\big|_{\partial\Omega} = 0$ for any such v this is equivalent to saying that

$$\int_\Omega |\nabla u|^2 dx \le \int_\Omega |\nabla(u + \varphi)|^2 dx$$

for all functions $\varphi \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\varphi\big|_{\partial\Omega}$. By density argument one may equally take $\varphi \in C_c^\infty(\Omega)$, i.e. φ is smooth with support compactly contained in Ω.

A similar argument to so-called elliptic operators in divergence form with $b^k = c = 0$ where we say L is in divergence form if for $u \in C^2(\Omega)$

$$Lu := \sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u).$$

Now $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu = 0$ in Ω and $u\big|_{\partial\Omega} = g$ if and only if it is the minimizer

$$E^{A,\Omega} : v \mapsto \int_\Omega \sum_{i,j=1}^n a^{ij}(x)\partial_i u(x)\partial_i u(x) dx$$

among all function $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying $v\big|_{\partial\Omega} = u\big|_{\partial\Omega} = g$. Again this is equivalent to saying that

$$\int_\Omega \sum_{i,j=1}^n a^{ij}(x)\partial_i u(x)\partial_i u(x) dx \le \int_\Omega \sum_{i,j=1}^n a^{ij}(x)\partial_i(u+\varphi)(x)\partial_i(u+\varphi)(x) dx$$

for all functions $\varphi \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\varphi\big|_{\partial\Omega}$ (resp. $\varphi \in C_c^\infty(\Omega)$).

Remark (Operators in divergence form). If $a^{ij} \in C^1(\Omega)$ then via product rule a divergence operator can be brought into non-divergence form. Note that in general the b^k-terms are non-zero and the arguements of this section require b^k and c to be zero, see however the section on the Lax–Milgram Theorem for more general results.

Observe that for $u_1, u_2 \in C^1(\Omega)$ and $\lambda \in (0,1)$ it holds

$$E^{A,\Omega}((1-\lambda)u_1 + \lambda u_2) = (1-\lambda)E^{A,\Omega}(u_1) + \lambda E^{A,\Omega}(u_2) - (1-\lambda)\lambda E^{A,\Omega}(u_1 - u_2).$$

Thus $E^{A,\Omega}$ is a convex function. Furthermore, if $\lambda : \Omega \to (0,\infty)$ is the ellipticity constant of L then

$$E^{A,\Omega}(u_1 - u_2) \ge \int \lambda(x)|\nabla(u_1 - u_2)(x)|^2 dx > 0$$

unless $u_1 = u_2 + c$ for a constant $c \in \mathbb{R}$. Thus the convexity inequality above is strict if the difference of u_1 and u_2 is non-constant.

The above argument shows that we should look at minimizers of $E^{A,\Omega}$. In particular, we have to show that the set

$$\{u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \,|\, E(u) = \inf_{\varphi \in C_c^\infty(\Omega)} E(u + \varphi)\}$$

is non-empty. Given a function u_0 with $u_0\big|_{\partial\Omega} = g$ we can always find a sequence u_n with $u_n\big|_{\partial\Omega}$ such that $\lim_{n\to\infty} E(u_n) = \inf_{\varphi\in C_c^\infty(\Omega)} E(u_0+\varphi)$. However, the sequence $(u_n)_{n\in\mathbb{N}}$ might not contain any convergent subsequence (in $C^2(\Omega)\cap C^0(\bar\Omega)$) as the bounds of $E(u_n)$ do not imply any bounds on the first and second derivatives[5].

To circumvent we may observe that $E^{A,\Omega}$ satisfies the parallelogram inequality. Hence it seems natural to look at

$$\Phi : u \mapsto \left(\int_\Omega \sum_{i,j=1}^n a^{ij}(x)\partial_i u(x)\partial_i u(x)dx\right)^{\frac{1}{2}}$$

this mapping from $\{u \in C^1(\Omega)\,|\,\Phi(u) < \infty\}$ to $[0,\infty)$ satisfies all properties of a norm but the definiteness. Indeed, as observed above if $u_1 = u_2 + c$ for a constant c then $\Phi(u_1) = \Phi(u_2)$. A natural choice is to add the L^2-norm of u and define

$$\Psi : u \mapsto \left(\int_\Omega |u|^2 dx + \int_\Omega \sum_{i,j=1}^n a^{ij}(x)\partial_i u(x)\partial_i u(x)dx\right)^{\frac{1}{2}}.$$

This mapping gives a norm on the vector space $Y = \{u \in C^1(\Omega)\,|\,\Psi(u) < \infty\}$. Thus we can take the completion of (Y,Ψ) to obtain a natural Banach spaces. The only problem with this construction is that the completion might depend on $(a^{ij} : \Omega \to \mathbb{R})_{i,j=1}^n$. As we are only interested in existence of minimizer in the Banach space we may replace the norm by a more suitable one.

Definition 4.7 (uniformly equivalent)**.** A two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are said to be uniformly equivalent if there is a constant $C \geq 1$ such that for all $v \in X$ it holds

$$C^{-1}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1.$$

The concept of uniformly equivalent shows that a Cauchy sequence with respect to one of the norms is also a Cauchy sequence of the other one. Hence we obtain the following lemma.

Lemma 4.8. *Given any two uniformly equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_1$ on a vector space X the completion with respect to either of the norm will give the same completion $\tilde X$ and the naturally associated norm $\|\cdot\|_1$ and $\|\cdot\|_2$ are still uniformly equivalent norms on $\tilde X$. In particular, they induce the same complete metric topology on $\tilde X$.*

Note that for each element $v \in \tilde X$ and every $v_n \to v$ with $v_n \in i(X)$ where i is the natural embedding of the completion process it holds $\|v\|_i = \lim \|v\|_i$.

Recall that the elliptic operator is *uniformly elliptic* if there are constant $\lambda, \Lambda \in (0,\infty)$ such that

$$\lambda\langle\xi,\xi\rangle \leq \langle\xi, A(x)\xi\rangle \leq \Lambda\langle\xi,\xi\rangle.$$

Thus for a uniformly elliptic operator we may show that

$$C^{-1}\Psi(u) \leq \|u\|_{W^{1,2}} \leq C\Psi(u)$$

[5]no obvious bound

where $C^2 = \max\{\lambda^{-1}, \Lambda\}$ and

$$\|u\|_{W^{1,2}} = \left(\int_\Omega |u|^2 dx + \int_\Omega \sum_{i=1}^n (\partial_i u)^2 dx\right)^{\frac{1}{2}}.$$

Thus the two norms are equivalent. We call $\|\cdot\|_{W^{1,2}}$ the ($W^{1,2}$-)Sobolev norm (sometimes also H^1-norm).

Definition 4.9 (First Sobolev space)**.** The completion of

$$\{u \in C^1(\Omega) \,|\, \|u\|_{W^{1,2}} < \infty\}$$

with respect to the norm $\|\cdot\|_{W^{1,2}}$ is called the first (L^2-)Sobolev space and is denoted by $W^{1,2}(\Omega)$.

Since the Sobolev norm is equivalent to the norm Ψ we can also extend the functional $E^{A,\Omega}$ to all function[6] in $W^{1,2}(\Omega)$. Indeed, it holds $u_n \to u$ in $W^{1,2}(\Omega)$ then $\|u_n - u\|_2 \to 0$ and $\Psi(u_n) \to \Psi(u)$. Thus observing that

$$E^{A,\Omega}(u) = \Psi(u)^2 - \|u\|_2^2$$

for $u \in C^1(\Omega) \cap W^{1,2}(\Omega)$ we can choose any $u_n \in C^1(\Omega) \cap W^{1,2}(\Omega)$ with $u_n \to u \in W^{1,2}(\Omega)$ and uniquely define

$$E^{A,\Omega}(u) = \lim_{n\to\infty} \Psi(u_n) - \|u_n\|_2^2.$$

Alternatively we may observe the following.

Lemma 4.10. *There is an isometric*[7] *embedding*

$$i : W^{1,2}(\Omega) \to \bigotimes_{i=0}^n L^2(\Omega)$$

satisfying

$$i(u) = (u, \partial_1 u, \dots, \partial_n u)$$

for all $u \in C^1(\Omega) \cap W^{1,2}(\Omega)$.

Proof. Just observe that i on $C^1(\Omega) \cap W^{1,2}(\Omega)$ is an isometry which extends uniquely to its closure which is by definition $W^{1,2}(\Omega)$. □

Thus we see that for $u \in C^1(\Omega) \cap W^{1,2}(\Omega)$ it holds $\partial_i u \in C^0(\Omega) \cap L^2(\Omega)$ so that for any $u \in W^{1,2}(\Omega)$ there is a uniquely defined object $\partial_i u \in L^2(\Omega)$ such that for all $u_n \to u$ in $W^{1,2}(\Omega)$ with $u_n \in C^1(\Omega) \cap W^{1,2}(\Omega)$ it holds $\partial_i u_n \to \partial_i u$ in $L^2(\Omega)$ (IMPORTANT: Currently $\partial_i u$ for $u \in W^{1,2}(\Omega)$ is just a suggestive notation. Later we will show that $\partial_i u$ satisfies indeed the properties of a partial derivative hence justifying the use of ∂_i). As uniform ellipticity implies that a^{ij} is bounded in Ω the functional $E^{A,\Omega}$ satisfies

$$E^{A,\Omega}(u) = \int_\Omega \sum_{i,j=1}^n a^{ij}(x)\partial_i u(x)\partial_i u(x)dx.$$

Another observation is the following.

[6]As $W^{1,2}(\Omega)$ behaves similar to $L^2(\Omega)$ the statement $u \in W^{1,2}(\Omega)$ is a function is meant to say that the measurable function u represents an equivalence class (of functions) in $W^{1,2}(\Omega)$.

[7]norm-preserving

Corollary 4.11. *The space $W^{1,2}(\Omega)$ is a Hilbert space. In particular, it is 2-uniformly convex.*

Now we have defined $E^{A,\Omega}$ on an appropriate Banach space. The next step is to find a minimizer given certain boundary data. As we currently do not have “trace operator” which are maps from $W^{1,2}(\Omega)$ to $L^2(\Omega)$, we try to find a $u \in W^{1,2}(\Omega)$ that is minimal among all perturbations φ with zero boundary data.

Definition 4.12 (Soblev space with zero boundary data)**.** The space $W_0^{1,2}(\Omega)$ is defined as the closure of $C_c^1(\Omega) \cap W^{1,2}(\Omega)$ in $W^{1,2}(\Omega)$.

It is easy to see that $(W_0^{1,2}(\Omega), \|\cdot\|_{W^{1,2}})$ is a closed subspace of the Banach space $W^{1,2}(\Omega)$ and thus a Banach space as well. Furthermore, for each $v \in W_0^{1,2}(\Omega)$ there is a sequence $v_n \in C_c^1(\Omega) \cap W^{1,2}(\Omega)$ with $v_n \to v$ in $W^{1,2}(\Omega)$. In particular, by continuity of $E^{A,\Omega}$ we see that

$$E^{A,\Omega}(u) \le E^{A,\Omega}(u+v) \quad \text{for all } v \in W_0^{1,2}(\Omega)$$

if and only if

$$E^{A,\Omega}(u) \le E^{A,\Omega}(u+\varphi) \quad \text{for all } \varphi \in C^1(\Omega) \cap W_0^{1,2}(\Omega).$$

Thus given $u_0 \in W^{1,2}(\Omega)$ we may change the minimization problem to find $v \in W_0^{1,2}(\Omega)$ such that

$$E_{u_0} : v \mapsto E^{A,\Omega}(u_0 + v).$$

Lemma 4.13. *For each $\epsilon \ge 0$ the sets*

$$C_\epsilon = \{v \in W_0^{1,2}(\Omega) \,|\, E_{u_0}(v) \le \inf E_{u_0} + \epsilon\}$$

is closed and convex with $C_\epsilon \subset C_{\epsilon'}$ for $\epsilon' \le \epsilon$. Furthermore, $C_\epsilon \ne \varnothing$ for $\epsilon > 0$ and each element in C_0 is a minimizer of E_{u_0}.

Proof. Convexity follows by observing that

$$\begin{aligned} E_{u_0}((1-\lambda)v_1 + \lambda v_2) &= E^{A,\Omega}((1-\lambda)(u_0+v_1) + \lambda(u_0+v_2)) \\ &\le (1-\lambda)E^{A,\Omega}(u_0+v_1) + \lambda E^{A,\Omega}(u_0+v_2) \\ &= (1-\lambda)E_{u_0}^{A,\Omega}(v_1) + \lambda E_{u_0}^{A,\Omega}(v_2). \end{aligned}$$

Furthermore, if $v_n \to v$ in $W_0^{1,2}(\Omega)$ then $u_0 + v_n \to u_0 + v$ in $W^{1,2}(\Omega)$. Thus continuity of $E^{A,\Omega}$ implies that E_{u_0} is continuous thence

$$C_\epsilon = E_{u_0}^{-1}((-\infty, \epsilon])$$

is closed. □

If C_ϵ is bounded then we would be allowed to use Corollary 4.5 to show that

$$C_0 = \bigcap_{n \in \mathbb{N}} C_{\frac{1}{n}} \ne \varnothing$$

which yields the existence of a minimizer for E_{u_0}.

However, in general we cannot ensure that $\|\cdot\|_{W^{1,2}}$ is bounded on C_ϵ as we have only the following for each $v \in C_\epsilon$

$$\begin{aligned}\|v\|^2_{W^{1,2}} &\le 2\|u_0+v\|^2_{W^{1,2}} + 2\|u_0\|^2_{W^{1,2}}\\ &\le 4\int_\Omega |u_0|^2 + 4\int_\Omega |v|^2dx + 2\Lambda E_{u_0}(v) + 2\|u_0\|^2_{W^{1,2}}\\ &\le 6\|u_0\|^2_{W^{1,2}} + 2\Lambda(\inf E_{u_0}+\epsilon) + \int_\Omega |v|^2dx.\end{aligned}$$

Thus in order to ensure that C_ϵ is bounded in $W^{1,2}_0(\Omega)$ it suffices to bound the $L^2(\Omega)$.

Surprisingly the following statement holds:

Theorem 4.14 (Weak version of Gagliardo–Nirenberg Sobolev inequality)**.** *For all $v \in W^{1,2}_0(\Omega)$ there is a constant $C = C(n)$ such that*

$$\int_\Omega |v|^2dx \le C\int_\Omega \sum_{i=1}^n |\partial_i v|^2dx.$$

Before we prove this theorem we want to show that C_ϵ is bounded. Indeed it holds

$$\begin{aligned}\|v\|^2_{W^{1,2}} &\le 6\|u_0\|^2_{W^{1,2}} + 2\Lambda(\inf E_{u_0}+\epsilon) + \int_\Omega |v|^2dx.\\ &\le 6\|u_0\|^2_{W^{1,2}} + 2\Lambda(\inf E_{u_0}+\epsilon) + C\int_\Omega \sum_{i=1}^n |\partial_i v|^2dx.\\ &\le 6\|u_0\|^2_{W^{1,2}} + 2\Lambda(\inf E_{u_0}+\epsilon) + 2CE_{u_0}(v) + 2C\|u_0\|^2_{W^{1,2}}\\ &\le (6+2C)\|u_0\|^2_{W^{1,2}} + (2\Lambda+2C)(\inf E_{u_0}+\epsilon)\end{aligned}$$

which implies that C_ϵ is bounded.

Corollary 4.15. *If L is an uniformly elliptic operator then for any $u_0 \in W^{1,2}(\Omega)$ there is a unique minimizer of the function $E_{u_0} = E^{A,\Omega}(u_0+\cdot)$.*

Proof. The existence follows from 4.5 using boundedness of C_ϵ implied by Gagliardo–Nirenberg. To observe uniqueness, assume $E_{u_0}(v) = E_{u_0}(v')$ for $v, v' \in C_0$. It suffices to show that $\|v-v'\|_{W^{1,2}} = 0$.

By convexity of C_0 we see that $\frac{1}{2}v + \frac{1}{2}v' \in C_0$ so that the parallelogram identity for $E^{A,\Omega}$ yields

$$\lambda\int_\Omega \sum_{i=1}^n |\partial_i(v-v')|^2dx \le \int_\Omega a^{ij}\partial_i(v-v')\partial_j(v-v')dx = 0.$$

Since $v - v' \in W^{1,2}_0(\Omega)$ Gagliardo–Nirenberg implies

$$\int_\Omega |v-v'|^2dx \le C\int_\Omega \sum_{i=1}^n |\partial_i(v-v')|^2dx = 0$$

showing that $\|v-v'\|_{L^2} = 0$. Thus $\|v-v'\|_{W^{1,2}} = 0$ proving the claim. □

Instead of proving the weak version of Gagliardo–Nirenberg let us prove the following more general statement.

Definition 4.16 (First L^p-Sobolev spaces)**.** Let $p \in [1,\infty)$ and define a mapping on $C^1(\Omega)$ by

$$\|u\|_{W^{1,p}} = \left(\int |u|^p dx + \int \sum_{i=1}^n |\partial_i u|^p dx\right)^{\frac{1}{p}}.$$

Then the first L^p-Sobolev space $W^{1,p}(\Omega)$ is defined as the completion of the vector space $X = \{u \in C^1(\Omega) \,|\, \|u\|_{W^{1,2}(\Omega)}\}$ equiped with the norm $\|\cdot\|_{W^{1,p}}$. Similarly, $W_0^{1,p}(\Omega)$ is the closure of $C_c^1(\Omega) \cap W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$.

Remark. Again it is possible to isometrically embed $W^{1,p}(\Omega)$ into $\otimes_{i=1}^n L^p(\Omega)$ to obtain objects $\partial_i u \in L^p(\Omega)$. This embedding then shows that the norm $\|\cdot\|_p$ is p'-uniformly convex if $p \in (1,\infty)$ and $p' = \max\{2,p\}$.

Theorem 4.17 (Gagliardo–Nirenberg Sobolev inequality)**.** *Let $v \in W_0^{1,p}(\Omega)$. For all $p \in [1,n)$ there is a constant $C_p = C(n,p)$ such that*

$$\left(\int_\Omega |v|^{p^*} dx\right)^{\frac{1}{p^*}} \le C_p \left(\int_\Omega \sum_{i=1}^n |\partial_i v|^p dx\right)^{\frac{1}{p}}$$

where

$$p^* = \frac{np}{n-p} \quad \text{if } p \in [1,n).$$

In order to prove the theorem we need a couple of technical lemma. For notational purpose we denote by $\hat{x}_i$ the vector obtained from $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ via remove the i-th coordinate, i.e. $\hat{x}_i = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$.

Lemma 4.18. *Assume $F_i : \mathbb{R}^{n-1} \to [0,\infty)$ for $i = 1,\dots,n$ are bounded continuous*[8] *functions with compact support. Then*

$$\int_{\mathbb{R}^n} \Pi_{i=1}^n F_i(\hat{x}_i)^{\frac{1}{n-1}} dx \le \Pi_{i=1}^n \left(\int_{\mathbb{R}^{n-1}} F_i(y) dy\right)^{\frac{1}{n-1}}.$$

Proof. In the following $x \in \mathbb{R}^n$. Then we write $\int F_i(\hat{y}_i) dy_1$, $i > 1$ for

$$\int F_i(y_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) dy_1.$$

Similarly,

$$\int F_i(\hat{y}_i) dy_1 dy_2 = \int F_i(y_1, y_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_n) dy_1 dy_2.$$

Using this notation we observe that integrating over the first coordinate and using the generalized Hölder inequality applied to the last $n-1$ terms yields

$$\begin{aligned}\int \prod_{i=1}^n F_i(\hat{y}_i)^{\frac{1}{n-1}} dy_1 &= F_1(\hat{x}_1)^{\frac{1}{n-1}} \int \prod_{i=2}^n F_i(\hat{x}_i)^{\frac{1}{n-1}} dy_1 \\ &\le F_1(\hat{x}_1)^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int F_i(\hat{y}_i) dy_1\right)^{\frac{1}{n-1}}.\end{aligned}$$

[8] One may assume measurablilty using Fubini's theorem.

Integrating this inequality over the second inequality yields

$$\begin{aligned}\int\int\prod_{i=1}^n F_i(\hat{y}_i)^{\frac{1}{n-1}}dy_1dy_2 &= \left(\int F_2(\hat{y}_2)dy_1\right)^{\frac{1}{n-1}}\\ &\quad\cdot\int\left[F_1(\hat{y}_1)^{\frac{1}{n-1}}\cdot\Pi_{i\geq 3}^n\left(\int\int F_i(\hat{y}_i)dy_1\right)^{\frac{1}{n-1}}\right]dy_2\\ &\leq\left(\int F_2(\hat{y}_2)dy_1\right)^{\frac{1}{n-1}}\cdot\\ &\quad\cdot\left(\int F_1(\hat{y}_1)dy_2\right)^{\frac{1}{n-1}}\cdot\prod_{i\geq 3}^n\left(\int\int F_i(\hat{y}_i)dy_1dy_2\right)^{\frac{1}{n-1}}\end{aligned}$$

where we applied again the generalized Hölder inequality to the last $n-1$ terms. Then a similar argument with y_3 gives

$$\begin{aligned}\int\int\int\prod_{i=1}^n F_i(\hat{x}_i)^{\frac{1}{n-1}}dy_1dy_2dy_3 &\leq\left(\int\int F_1(\hat{y}_1)dy_2dy_3\right)^{\frac{1}{n-1}}\\ &\quad\cdot\left(\int\int F_2(\hat{y}_2)dy_1dy_3\right)^{\frac{1}{n-1}}\\ &\quad\cdot\left(\int\int F_3(\hat{y}_3)dy_1dy_2\right)^{\frac{1}{n-1}}\\ &\quad\cdot\prod_{i\geq 4}^n\left(\int\int\int F_i(\hat{y}_i)dy_1dy_2dy_3\right)^{\frac{1}{n-1}}\end{aligned}$$

where

$$I_k=\{(i_1,\dots,i_k)\in\{1,\dots,k\}\,|\,i_l\neq i_k\}.$$

Continuing with y_4 to y_n yields

$$\int\Pi_{i=1}^n F_i(\hat{y}_i)^{\frac{1}{n-1}}dy\leq\prod_{i=1}^n\left(\int F_i(\hat{y}_i)d\hat{y}_i\right)^{\frac{1}{n-1}}$$

which is the claim. □

Lemma 4.19. *For $v\in C_c^1(\Omega)\cap W^{1,1}(\Omega)$ it holds*

$$\left(\int_\Omega|v|^{\frac{n}{n-1}}dx\right)^{\frac{n-1}{n}}\leq\int_\Omega\sum_{i=1}^n|\partial_i v|dx.$$

Proof. For $x=(x_1,\dots,x_n)\in\Omega$ and $i=1,\dots,n$ it holds

$$|u(x)|\leq\int_{-\infty}^{x_i}|\partial_i u|(x_1,\dots,y_i,\dots,x_n)dy_i$$

so that

$$|u(x)|^{\frac{n}{n-1}}\leq\left(\Pi_{i=1}^n\int|\partial_i u|(\hat{y}_i)dy_i\right)^{\frac{1}{n-1}}.$$

Define now $F_i(\hat{x}_i) = \int |\partial_i u(\hat{y}_i)| dy_i$. Then the previous lemma shows

$$\int |u(x)|^{\frac{n}{n-1}} dx \leq \int \prod_{i=1}^{n} F_i(\hat{x}_i)^{\frac{1}{n-1}} dx \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} F_i(y) dy \right)^{\frac{1}{n-1}}.$$

Observing that

$$\int_{\mathbb{R}^{n-1}} F_i(y) dy \leq \int_{\Omega} \sum_{j=1}^{n} |\partial_j u| dx$$

shows that

$$\int |u(x)|^{\frac{n}{n-1}} dx \leq \left(\int_{\Omega} \sum_{j=1}^{n} |\partial_j u| dx \right)^{\frac{n}{n-1}}$$

proving the result. □

Proof of the theorem. By density it suffices to show the result for functions $C_c^1(\Omega) \cap W^{1,p}(\Omega)$. The previous lemma shows that it is true for $p = 1$ and with $C(n,1) = 1$.

For $p \in (1, n)$ and $\gamma > 0$ observe that $|u|^\gamma \in C_c^1(\Omega)$ whenever $u \in C_c^1(\Omega)$. Furthermore, it holds $|\partial_i |u|^\gamma| = \gamma |u|^{\gamma-1} |\partial_i u|$.

Thus using the result for $p = 1$ applied to $|u|^\gamma$ gives

$$\begin{aligned}
\left(\int |u|^{\frac{\gamma n}{n-1}} \right)^{\frac{1}{n-1}} &\leq \int \sum_{j=1}^{n} |\partial_j |u|^\gamma| dx \\
&= \int \gamma |u|^{\gamma-1} \sum_{j=1}^{n} |\partial_j u| dx \\
&\leq \gamma \left(\int |u|^{(\gamma-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int \sum_{j=1}^{n} |\partial_j u|^p dx \right)^{\frac{1}{p}}.
\end{aligned}$$

If we now choose γ such that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p-1}{p}$ then $\gamma = \frac{n-1}{n-p} p > 0$ and $p^* = \frac{np}{n-p} = \frac{\gamma n}{n-p}$ so that

$$\left(\int |u|^{p^*} \right)^{\frac{n-1}{n}} \leq \gamma \left(\int |u|^{p^*} \right)^{\frac{p-1}{p}} \left(\int \sum_{j=1}^{n} |\partial_j u|^p dx \right)^{\frac{1}{p}}.$$

To conclude observe that

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{(n-1)p - n(p-1)}{pn} = \frac{n-p}{np} = \frac{1}{p^*}$$

proving

$$\left(\int |u|^{p^*} \right)^{\frac{1}{p^*}} = \left(\int |u|^{p^*} \right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \gamma \left(\int \sum_{j=1}^{n} |\partial_j u|^p dx \right)^{\frac{1}{p}}.$$

□

Below we will provide a proof via embedding theorems of the Riesz potential operator whose proof depends on the Hardy–Littlewood Maximal Functional Theorem.

4.5. **Weak derivatives.** From the definition of $\partial_i u$ as the i-th coordinate of the natural embedding $i : W^{1,p}(\Omega) \hookrightarrow \bigotimes_{i=0}^n L^p(\Omega)$ we see that $\partial_i(\alpha u + \beta v) = \alpha\partial_i u + \beta\partial_i v$ and $\|\partial_i u\|_p \le \|u\|_{W^{1,p}}$. Thus $\partial_i : u \mapsto \partial_i u$ is a bounded linear operator from $W^{1,p}(\Omega)$ into $L^p(\Omega)$.

Lemma 4.20. *For all $u \in W^{1,p}(\Omega)$ and all $\varphi \in C_c^1(\Omega)$ it holds*

$$\int_\Omega u \cdot \partial_i \varphi dx = -\int \partial_i u \cdot \varphi dx.$$

Proof. Observe that the result holds for all $u \in W^{1,p}(\Omega) \cap C^1(\Omega)$. If $u_n \to u$ in $W^{1,p}(\Omega)$ for $u_n \in C^1(\Omega) \cap W^{1,p}(\Omega)$ then $u_n \to u$ and $\partial_i u_n \to \partial_i u$ in $L^2(\Omega)$ which implies

$$\begin{aligned}\int_\Omega u \cdot \partial_i \varphi dx &= -\lim_{n\to\infty} \int u_n \cdot \partial_i \varphi dx \\ &= -\lim_{n\to\infty} \int \partial_i u_n \cdot \varphi dx = -\int \partial_i u \cdot \varphi dx.\end{aligned}$$

□

Definition 4.21 (Weak derivative)**.** A function $g_i \in L^1_{loc}(\Omega)$ is called a *weak i-th coordinate derivative* of $u \in L^1_{loc}(\Omega)$ for some $i \in \{1, \ldots, n\}$ if for all $\varphi \in C_c^1(\Omega)$ it holds

$$\int_\Omega g_i \cdot \varphi dx = \int u \cdot \partial_i \varphi dx.$$

Remark. By abuse of notation the index i of g_i denotes also "which" derivative is to be considered.

It is not difficult to see that weak derivatives are unique. Furthermore, if $u \in W^{1,p}(\Omega)$ then $\partial_i u$ is a weak coordinate derivative of u. Furthermore, if $u = v$ on some subset $\Omega' \subset\subset \Omega$ and g_i and h_i are weak derivatives of u and resp. v then $g_i = h_i$ on Ω'. In particular, if u has compact support then any weak derivatives has compact support as well.

Lemma 4.22 (Chain rule)**.** *For all $u \in W^{1,p}(\Omega)$ and $\alpha \in C^1(\mathbb{R})$ with $\alpha(0) = 0$ and $|\alpha'| \le M$ for some $M > 0$ it holds $\alpha(u) \in L^p(\Omega)$ and $\alpha'(u) \cdot \partial_i u$ is a weak derivative of $\alpha(u)$.*

Proof. The result holds for function $u \in C^1(\Omega)$. If $u_n \to u$ in $W^{1,p}(\Omega)$ for $u_n \in C^1(\Omega) \cap W^{1,p}(\Omega)$ then $u_n \to u$ and $\partial_i u_n \to \partial_i u$ in $L^2(\Omega)$. In particular, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ and $\partial_i u_{n_k} \to \partial_i u$ almost everywhere.

Now

$$\|\alpha(u_n) - \alpha(u)\|_p \le M\|u_n - u\|_p$$

implies that $(\alpha(u_{n_k}))_{n\in\mathbb{N}}$ is a Cauchy in $L^p(\Omega)$ converging to $\alpha(u)$. Note that $\alpha'(u) \cdot \partial_i u$ is clearly in $L^p(\Omega)$. Thus we only need to show $\alpha'(u) \cdot \partial_i u$ is a weak derivative. For this let $\varphi \in C_c^1(\Omega)$ then by the Dominated Convergence Theorem it holds

$$\begin{aligned}\int \alpha'(u) \cdot \partial_i u \cdot \varphi dx &= \lim_{k\to\infty} \int \alpha'(u_{n_k}) \cdot \partial_i u_{n_k} \cdot \varphi dx \\ &= \lim_{k\to\infty} \int \partial_i \alpha(u_{n_k}) \cdot \varphi dx \\ &= -\lim_{k\to\infty} \int \alpha(u_{n_k}) \cdot \partial_i \varphi dx = -\int \alpha(u) \cdot \partial_i \varphi dx.\end{aligned}$$

□

Lemma 4.23 (Product rule)**.** *If $\zeta \in C^1(\Omega)$ with $\|\zeta\|_{C^1} < \infty$ and $g \in L^p(\Omega)$ is a weak derivative of $u \in L^p(\Omega)$ then $h_i = \partial_i \zeta \cdot u + \zeta \cdot g \in L^p(\Omega)$ is a weak derivative of $\zeta \cdot u$.*

Remark. The result holds more generally for $u \cdot v$ for general $u, v \in L^\infty(\Omega) \cap L^p(\Omega)$ admitting weak derivatives.

Proof. Let $\varphi \in C_c^1(\Omega)$ then $\zeta \cdot \varphi \in C_c^1(\Omega)$ so that

$$\begin{aligned} -\int \zeta \cdot u \cdot \partial_i \varphi dx &= -\int u \cdot \partial_i(\zeta \cdot \varphi) dx + \int \partial_i \zeta \cdot u \cdot \varphi dx \\ &= \int \zeta \cdot g \cdot \varphi dx + \int \partial_i \zeta \cdot u \cdot \varphi dx. \end{aligned}$$

□

Using the weak topology of $W^{1,p}(\Omega)$ and the fact that $\|\alpha(u)\|_{W^{1,p}} \leq M\|\alpha(u)\|$ for all $u \in C^1(\Omega)$ we may even show that $\alpha(u) \in W^{1,p}(\Omega)$. Instead of this we show the following theorem.

Theorem 4.24. *A measurable function $u : \Omega \to \mathbb{R}$ is in $W^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and for each $i \in \{1, \cdots, n\}$ there is a weak derivative $g_i \in L^p(\Omega)$.*

Note that $u \in W^{1,p}(\Omega)$ if and only if it can be $\|\cdot\|_{W^{1,p}}$-approximated by C^1-functions. The theorem is saying that L^p-functions with weak derivative in L^p always admit C^1-approximation. This was an open question until Meyers and Serrin proved in a paper titled "$H = W$" in 1964.

In order to prove the theorem we need the concepts of cut-off functions, mollifiers and smooth partitions of unit. For this define the following function

$$\varphi(x) = \begin{cases} \frac{1}{c_n} e^{-\frac{1}{1-\|x\|^2}} & \|x\| < 1 \\ 0 & \|x\| \geq 1 \end{cases}$$

where

$$c_n = \int_{B_1(\mathbf{0})} e^{-\frac{1}{1-\|x\|^2}} dx,$$

i.e. $\int_{\mathbb{R}^n} \varphi dx = 1$. Also define $\varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon)$.

Given a function $u \in L^1_{loc}(\mathbb{R}^n)$ define

$$u_\epsilon(x) = \int_{\mathbb{R}^n} \varphi_\epsilon(x-y) u(y) dy.$$

Proposition 4.25. *For all $\epsilon > 0$ the functions u_ϵ are $C^\infty(\mathbb{R}^n)$ and*

$$\operatorname{supp} u_\epsilon \subset \operatorname{supp} u + \operatorname{supp} \varphi_\epsilon = \operatorname{supp} u + B_\epsilon(\mathbf{0}).$$

Furthermore,

$$\|u_\epsilon\|_p \leq \|u\|_p.$$

Finally if $g_i \in L^p(\Omega)$ is a weak derivative then $(g_i)_\epsilon = \partial_i u_\epsilon$.

Proof. The first two statements follows by exchanging integration and differentiation.

The next claim follows from the fact that

$$\begin{aligned}|u_\epsilon(x)|^p &= \left|\int \varphi_\epsilon(x-y)u(y)dx\right|^p \\ &\le \int \varphi_\epsilon(x-y)|u(y)|^p dy.\end{aligned}$$

Thus

$$\begin{aligned}\int |u_\epsilon|^p dx &\le \int\int \varphi_\epsilon(y)|u(x-y)|^p dydx \\ &\le \int \varphi_\epsilon(y)\int |u(x-y)|^p dxdy \\ &= \int \varphi_\epsilon(y)\int |u|^p dxdy = \int |u|^p dx.\end{aligned}$$

Furthermore, the last claim let $\psi \in C_c^1(\mathbb{R}^n)$. Then again using Fubini

$$\begin{aligned}\int (g_i)_\epsilon \cdot \psi dx &= \int \varphi_\epsilon(y)\int g_i(x-y)\psi(x)dxdy \\ &= \int \varphi_\epsilon(y)\int u(x-y)\partial_i\psi(x)dxdy \\ &= \int u_\epsilon \cdot \partial_i\psi dx.\end{aligned}$$

□

Lemma 4.26. *If $u \in L^p(\mathbb{R}^n)$ has compact support then*

$$\|u_\epsilon - u\|_p \to 0 \quad \textit{as } \epsilon \to 0.$$

Proof. Let $u_n \to u$ in $L^p(\mathbb{R})$ such that $u_n \in C_c^0(\Omega)$ where Ω is a bounded open subset with $\operatorname{supp} u \subset \Omega$. It is easy to see that $(u_n)_\epsilon \to u_n$ uniformly in Ω. For a given $\delta > 0$ choose $n \ni \mathbb{N}$ such that

$$\|u - u_n\|_p \le \delta.$$

Now it holds

$$\begin{aligned}\lim_{\epsilon\to 0}\|u_\epsilon - u\|_p &\le \lim_{\epsilon\to 0}[\|u_\epsilon - (u_n)_\epsilon\|_p + \|(u_n)_\epsilon - u_n\|_p + \|u_n - u\|_p] \\ &\le \lim_{\epsilon\to 0}[\|(u_n)_\epsilon - u_n\|_p + 2\|u_n - u\|_p] \le 2\delta.\end{aligned}$$

Since $\delta > 0$ is arbitrary we see that $\|u_\epsilon - u\|_p \to 0$ as $\epsilon \to 0$. □

Corollary 4.27. *For all $u \in L^p(\mathbb{R}^n)$ with compact support admitting weak derivatives $g_i \in L^p(\mathbb{R}^n)$ for $i = 1, \dots, n$ there is a sequence $u_n \in C_c^1(\mathbb{R}^n)$ such that $u_n \to u$ and $\partial_i u_n \to g_i$ in $L^p(\mathbb{R}^n)$. In particular, $u \in W^{1,p}(\mathbb{R}^n)$.*

In order to prove the theorem we need to split a given Sobolev function into summands with compact support in Ω.

Definition 4.28 (Partition of Unitity)**.** Let Ω be open and $\{\Omega_n\}_{n\in\mathbb{N}}$ be an locally finite covering of bounded sets. A family of function $\eta_n \in C_c^\infty(\Omega_i)$ is called a smooth partition of unity subordinate to $\{\Omega_n\}_{n\in\mathbb{N}}$ if

$$\sum_{n\in\mathbb{N}} \eta_n \equiv 1.$$

Remark. Note that local finiteness of the covering implies that for each $x \in \Omega$ there are finitely many $n_1^x, \dots, n_{m(x)}^x \in \mathbb{N}$ such that $x \in \Omega_{n_k^x}$. In particular, $\sum \eta_n(x) = \sum_{k=1}^{m(x)} \eta_{n_k^x}(x)$.

Proposition 4.29. *For each locally finite bounded covering $\{\Omega_n\}_{n\in\mathbb{N}}$ of an open set Ω with $\Omega_n \subset\subset \Omega$ there is a smooth partition of unity subordinate to the covering.*

Proof. Given a locally finite covering we can finite open set $V_n \subset\subset \Omega_n$ such that $\{V_n\}_{n\in\mathbb{N}}$ is still a covering of Ω. Just observe that given any covering $\{U_n\}_{n\in\mathbb{N}}$ of Ω with $U_n \subset\subset \Omega$ and $n_0 \in \mathbb{N}$ there is an $\epsilon > 0$ such that

$$\{U_{n_0}^{-\epsilon}\} \cup \{U_n\}_{n\neq n_0}$$

is still a locally finite covering where

$$U_{n_0}^{-\epsilon} = \{x \in U_{n_0} \mid d(x, \partial U_{n_0}) > \epsilon\}.$$

Thus the sets $V_n = \Omega_n^{-\epsilon_n}$ can be constructed inductively.

Define now

$$u_n := (\chi_{V_n})_{\frac{\epsilon_n}{2}}$$

and

$$\eta_n = \frac{u_n}{\sum_{m\in\mathbb{N}} u_m}.$$

By construction, $u_n \in C_c^\infty(\Omega_n)$ and the summand in the definition of η_n consists of only finitely many u_m. Furthermore,

$$\sum_{n\in\mathbb{N}} \eta_n(x) = \frac{\sum_{n\in\mathbb{N}} u_n(x)}{\sum_{m\in\mathbb{N}} u_m(x)} = 1.$$

□

Proof of the Theorem **??**. It suffices to show that any function $u \in L^p(\Omega)$ with weak derivatives g_i can be approximated by C^1-functions u_n such that $\partial_i u_n$ converges to g_i.

Define

$$\Omega_n = \{x \in \Omega \mid d(x, \partial\Omega) \in (\frac{1}{n-1}, \frac{1}{n+1})\}.$$

Then $\{\Omega_n\}_{n\in\mathbb{N}}$ is a locally finite covering of Ω and each Ω_n is bounded. Thus there is a partition of unit η_n. Observe now that for sufficiently small ϵ_n the function $(\eta_n \cdot u)_{\epsilon_n}$ is in $C_c^\infty(\cup_{k=1}^{n+1} \Omega_k)$ and

$$\|(\eta_n \cdot u)_{\epsilon_n} - \eta_n \cdot u\|_p, \|(\partial_i(\eta_n \cdot u))_{\epsilon_n} - h_i^n\|_p \le \frac{\delta}{2^n}.$$

where $h_i^n = \partial_i \eta_n \cdot u + \eta_n \cdot g_i \in L^p(\Omega)$ is the weak derivative of $\eta_n \cdot u$. Observe that $g_i = \sum_{n\in\mathbb{N}} h_i^n$.

Thus $u_\delta = \sum_{n\in\mathbb{N}} (\eta_n \cdot u)_{\epsilon_n}$ is well-defined and in $C^\infty(\Omega)$ and it holds

$$\|u_\delta - u\|_p \le \sum_{n\in\mathbb{N}} \|(\eta_n \cdot u)_{\epsilon_n} - \eta_n \cdot u\|_p \le \delta \sum_{n\in\mathbb{N}} 2^{-n} = \delta$$

$$\|\partial_i u_\delta - g_i\|_p \le \sum_{n\in\mathbb{N}} \|\partial_i(\eta_n \cdot u)_{\epsilon_n} - h_i\|_p \le \delta \sum_{n\in\mathbb{N}} 2^{-n} = \delta$$

Letting $\delta \to 0$ proves the result. □

4.6. **Higher order Sobolev spaces.**

Definition 4.30 (Higher order Sobolev spaces)**.** Let $u \in C^k(\Omega)$ then we can define

$$\|u\|_{W^{k,p}} = \left(\sum_{|I| \le k} \int_\Omega |\partial_I u|^p dx \right)$$

where $I = (i_1, \cdots, i_k)$ is a multi-index and $\partial_I u = \partial_{i_1 \cdots i_n} u$ and if $|I| = 0$ then $\partial_I u = u$.

The k-th order Sobolev space $W^{k,p}(\Omega)$ is the completion of $\{u \in C^k(\Omega) \,|\, \|u\|_{W^{k,p}} < \infty\}$ with respect to $\|\cdot\|_{W^{k,p}}$.

Similarly, the space $W_0^{k,p}(\Omega)$ is the closure of $C_c^k(\Omega) \cap W^{k,p}(\Omega)$ in $W^{k,p}(\Omega)$.

Remark. Again there is a natural embedding of $C^k(\Omega) \cap W^{k,p}(\Omega)$ into

$$\bigotimes_{i=0}^{N_k} L^p(\Omega)$$

where $N_k = \sum_{i=1}^k n^i$ which is defined by

$$u \mapsto (\partial_I u)_{|I| \le k}.$$

This extends naturally to $u \in W^{k,p}(\Omega)$ with well-defined objects $\partial_I u \in L^p(\Omega)$, $|I| \le k$.

Lemma 4.31. *Let $u \in W^{k,p}(\Omega)$. Then for all $|I| \le k$ and $\varphi \in C_c^k(\Omega)$ it holds*

$$\int \partial_I u \cdot \varphi dx = (-1)^{|I|} \int u \cdot \partial_I \varphi dx.$$

Definition 4.32 (Weak k-th derivatives)**.** Given a multi-index I with $k = |I| > 0$ we say $g_I \in L^1_{loc}(\Omega)$ is a weak k-th order derivative of $u \in L^1_{loc}(\Omega)$ if for all $\varphi \in C_c^k(\Omega)$ it holds

$$\int_\Omega g_I \cdot \varphi dx = (-1)^k \int_\Omega u \cdot \partial_I \varphi dx.$$

Remark. By abuse of notation, the index I of g_I denotes also “which” derivative is to be considered.

The following is an easy observation which follows from commutativity of the partial derivatives of C^k-functions.

Lemma 4.33. *Let $u \in W^{k,p}(\Omega)$ and I_1 and I_2 be two multi-indices such that the composed multi-index $I = I_1 \sqcup I_2 = (i_1^1, \ldots, i_l^1, i_1^2, \ldots i_m^2)$ satisfies $|I| \le k$. Then g_I is a weak m-th order derivative of g_{I_1} and a weak l-th order derivative of g_{I_2}.*

The following theorem can be proved almost exactly as Theorem 4.24.

Theorem 4.34. *Let $u \in L^p(\Omega)$ then the following are equivalent:*

- *$u \in W^{k,p}(\Omega)$, i.e. there is a sequence $u_n \in C^k(\Omega) \cap W^{k,p}(\Omega)$ with $u_n \to u$ in $W^{k,p}(\Omega)$.*
- *For each multi-index I with $|I| \le k$ there is a weak derivative $g_I \in L^p(\Omega)$ of u.*
- *$u \in W^{1,p}(\Omega)$ and $g_i \in W^{k-1,p}(\Omega)$ for each $i \in \{1, \ldots, n\}$.*

Note that if $u \in C_c^k(\Omega)$ then $\partial_I u \in C_c^{k-|I|}(\Omega)$ whenever $|I| \le k$. Thus if $u \in W_0^{k,p}(\Omega)$ then for each $|I| < k$ it holds $\partial_I u \in W_0^{k-|I|,p}(\Omega)$. Furthermore, $W_0^{l,p}(\Omega) \subset W_0^{1,p}(\Omega)$ Thus we can apply Theorem 4.17 to each $\partial_I u \in W^{k-|I|}(\Omega)$ and obtain the following

Theorem 4.35 (Higher order Gagliardo–Nirenberg)**.** *If $p \in [1,n)$ and $u \in W_0^{k,p}(\Omega)$ then $u \in W_0^{k-l,p^*}(\Omega)$ and*

$$\|u\|_{W^{k-1,p^*}} \le C_p \|u\|_{W^{k,p}}$$

where

$$p^* = \frac{np}{n-p}.$$

Corollary 4.36. *Let $l \in \{0,\dots,k-1\}$. If $p \in [1, \frac{n}{k-l})$ and $u \in W_0^{k,p}(\Omega)$ then for it holds $u \in W_0^{l,q}(\Omega)$ with*

$$\|u\|_{W^{l,q}} \le C_{p,(k-l)} \|u\|_{W^{k,p}}$$

where

$$q = \frac{np}{n-(k-l)p}.$$

Remark. For $p = 2$ and $\frac{n-2}{2} < (k-l) < \frac{n}{2}$ it holds

$$\frac{2n}{n-(k-l)2} > n.$$

4.7. **Poincaré inequality, and embeddings of Sobolev and Morrey.**

Proposition 4.37. *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$. Then for each $p \in [1,\infty)$ there for all $u \in W^{1,p}(\Omega)$ it holds*

$$⨍_\Omega |u-u_\Omega|^p dx \le 2^n (\operatorname{diam}\Omega)^p ⨍_\Omega \sum_{i=1}^n |\partial_i u|^p dx^{\frac{1}{p}}$$

where

$$u_\Omega = ⨍_\Omega u dx.$$

Proof. Set $g_u = (\sum_{i=1}^n |\partial_i u|^p)^{\frac{1}{p}}$ and $\ell = \operatorname{diam}\Omega$. Since C^1-functions are dense in $W^{1,p}(\Omega)$ it suffices to assume $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. By Jensen's inequality we have

$$\begin{aligned} \int_\Omega |u-u_\Omega|^p dx &= \int_\Omega \left| u(x) - ⨍_\Omega u dy \right|^p dx \\ &\le \int_\Omega ⨍_\Omega |u(x)-u(y)|^p \, dy dx. \end{aligned}$$

Let given $x, y \in \Omega$ define $\gamma_{xy}(t) = (1-t)x + ty$ and note by convexity of Ω we have $\gamma_{xy}(t) \in \Omega$. Thus

$$\begin{aligned} |u(x)-u(y)| &= \|x-y\| \left| \int_0^1 \partial_t u((1-t)x+ty) dt \right| \\ &\le \ell \int_0^1 g_u(\gamma_{xy}(t)) dt. \end{aligned}$$

Again using Jensen's inequality we get

$$\begin{aligned}\int |u-u_\Omega|^p dx &\le \int_\Omega \fint_\Omega |u(x)-u(y)|^p dx \\ &\le \ell^p \int_\Omega \fint_\Omega \left(\int_0^1 g_u(\gamma_{xy}(t))dt\right)^p dydx \\ &\le \ell^p \int_\Omega \fint_\Omega \int_0^1 g_u(\gamma_{xy}(t))^p dt^p dydx \\ &= \frac{2\ell^p}{|\Omega|}\int_\Omega\int_\Omega\int_{\frac12}^1 g_u(\gamma_{xy}(t))^p dtdydx\end{aligned}$$

where we use the symmetry $\gamma_{xy}(t)=\gamma_{yx}(1-t)$.

Using change of coordinates we also have

$$\int_\Omega g_u(\gamma_{xy}(t))^p dy = \frac{1}{t^n}\int_{\Omega_{t,x}} g_u(y)^p dy \le \frac{1}{t^n}\int_\Omega g_u(y)^p dy$$

where

$$\Omega_{t,x} = \{\gamma_{xy}(t)\,|\,y\in\Omega\}\subset\Omega.$$

We estimate

$$\int_{\frac12}^1 t^{-n}dt \le 2^{n-1}$$

so that

$$\begin{aligned}\int |u-u_\Omega|^p dx &\le \frac{2\ell^p}{|\Omega|}\int_\Omega\int_{\frac12}^1 \frac{1}{t^n}\int_\Omega g_u(y)^p dydtdx \\ &\le \frac{2^n\ell^p}{(n-1)|\Omega|}\int_\Omega\int_\Omega g_u(y)^p dydx = 2^n\ell^p\int_\Omega g_u(y)^p dy.\end{aligned}$$

Dividing each side by $|\Omega|$ gives the result. □

Observe that the Hölder inequality implies

$$\left(\fint_\Omega \sum_{i=1}^n |\partial_i u|dx\right) \le \left(\fint_\Omega \sum_{i=1}^n |\partial_i u|^p dx\right)^{\frac1p}$$

which yields the following corollary.

Corollary 4.38. *For all $p\in[1,\infty)$ it holds*

$$\fint_\Omega |u-u_\Omega|dx \le 2^n \operatorname{diam}\Omega \left(\fint_\Omega \sum_{i=1}^n |\partial_i u|^p dx\right)^{\frac1p}.$$

Lemma 4.39. *If Ω is convex then there are constants c_Ω and C_Ω such that for all $x\in\Omega$ and $r>0$ it holds*

$$c_\Omega\cdot r^n \le |B_r(x)\cap\Omega| \le C_\Omega\cdot r^n.$$

Proposition 4.40 (Morrey's Embedding)**.** *Assume Ω is bounded and convex. If $u\in C^1(\Omega)\cap W^{1,p}(\Omega)$ for $p\in(n,\infty)$ then $u\in C^{0,1-\frac{n}{p}}(\Omega)$ such that*

$$|u(x)-u(y)| \le C_n\|Du\|_p d(x,y)^{1-\frac{n}{p}}.$$

Proof. Let $x, y \in \Omega$ and for $i \in \mathbb{N}_{>0}$ define balls

$$B_i = B_{2^{-(i-1)}d(x,y)}(x) \cap \Omega$$
$$B_{-i} = B_{2^{-(i-1)}d(x,y)}(y) \cap \Omega$$

and

$$B_0 = B_{2d(x,y)}(x) \cap \Omega.$$

Then each ball B_i, $i \in \mathbb{Z}$ is convex and for $i > 0$ it holds

$$\operatorname{diam} B_i = 2^{-i} d(x,y)$$

and

$$|B_i| \geq c_\Omega \cdot 2^{-(i-1)n} d(x,y)^n \geq c_\Omega \cdot 2^{-in} d(x,y)^n.$$

Let $g_u = (\sum_{i=1}^n |\partial_i u|^p)^{\frac{1}{p}}$. Then

$$\begin{aligned}
|u_{B_i} - u_{B_{i+1}}| &= |\fint_{B_i} u dx - \fint_{B_{i+1}} u dy| \\
&\leq \fint_{B_{i+1}} |u - (\fint_{B_i} u dy)| dx \\
&\leq \fint_{B_{i+1}} |u - \fint_{B_i} u dy| dx \\
&\leq \fint_{B_i} |u - u_{B_i}| dx \\
&\leq 2^n (\operatorname{diam} B_i) \frac{1}{|B_i|^{\frac{1}{p}}} \left(\int_{B_i} g_u^p dx \right)^{\frac{1}{p}} \\
&\leq c_{p,n} \cdot 2^{-i(1-\frac{n}{p})} d(x,y)^{1-\frac{n}{p}} \|g_u\|_p
\end{aligned}$$

for $c_{n,p} = 2^n \cdot c_\Omega^{-\frac{1}{p}}$. Similarly,

$$|u_{B_{-i}} - u_{B_{-(i+1)}}| \leq c_{p,n} \cdot 2^{-i(1-\frac{n}{p})} d(x,y)^{1-\frac{n}{p}} \|g_u\|_p$$

and

$$|u_{B_{\pm 1}} - u_{B_0}| \leq 4 \cdot c_{n,p} \cdot d(x,y)^{1-\frac{n}{p}} \|g_u\|_p.$$

Since u is continuous we have

$$u_{B_i} \to u(x)$$

and

$$u_{B_{-i}} \to u(y).$$

Thus we get the telescope sum

$$\begin{aligned}
|u(x) - u(y)| &\leq \sum_{i \in \mathbb{Z}} |u_{B_i} - u_{B_{i+1}}| \\
&\leq 8 \cdot c_{p,n} \cdot \left[\sum_{i=0}^n 2^{-i(1-\frac{n}{p})} \right] \cdot \|g_u\|_p \cdot d(x,y)^{1-\frac{n}{p}} \\
&= C_{p,n} \cdot \|g_u\|_p \cdot d(x,y)^{1-\frac{n}{p}}
\end{aligned}$$

where

$$C_{p,n} = 8 \cdot c_{p,n} \cdot \left[\sum_{i=0}^n 2^{-i(1-\frac{n}{p})} \right] < \infty.$$

Since $\|g_u\|_p = \|Du\|_p$ we conclude. □

Note that for any Ω there is a ball B of radius $\frac{1}{2}\operatorname{diam}\Omega$ containing Ω. Since trivially any function $u \in W_0^{k,p}(\Omega)$ is also the $W^{1,p}$-limit of function $C_c^k(\Omega) \subset C^k(B)$ we have $u \in W^{k,p}(B)$.

Corollary 4.41. *If $u \in W_0^{k,p}(\Omega)$ with $p > n$ then $u \in C_0^{k-1,1-\frac{n}{p}}(\Omega)$ with*

$$\|u\|_{C^{k-1,1-\frac{n}{p}}} \le C_{p,n}\|u\|_{W^{k,p}}.$$

In combination with the Gagliardo–Nirgenberg we also obtain the following.

Corollary 4.42. *If $u \in W_0^{k,p}(\Omega)$ and $kp > n$ then $u \in C^{k-l,\gamma}$ where*

$$l = \left\lfloor \frac{n}{p} \right\rfloor + 1$$

and

$$\gamma = \begin{cases} l - \frac{n}{p} & \frac{n}{p} \notin \mathbb{N} \\ \text{any number in } (0,1) & \frac{n}{p} \in \mathbb{N}. \end{cases}$$

Remark. Observe that for $\frac{n-2}{2} < (k-(l+1))$ any Sobolev function $u \in W_0^{k,p}(\Omega)$ is in $C^l(\Omega)$ with bounded C^l-norm.

For completeness we also give a Sobolev inequality for general functions $u \in W^{1,p}(\Omega)$. We only sketch its proof as it suffices to have bounds for functions $u \in W_0^{1,p}(\Omega)$.

Proposition 4.43 (Sobolev Embedding)**.** *Assume Ω is bounded and convex. Then there is a $C = C(n,p) > 0$ such that for all $u \in W^{1,p}(\Omega)$ with $p \in [1,n)$ it holds*

$$\|u - u_\Omega\|_{p^*} \le C\|Du\|_p.$$

Note that this is a variant of the Sobolev inequality of Gagliardo–Nirenberg. Indeed, if $u \in W_0^{1,p}(\Omega)$ for some bounded Ω then there is a (convex) ball $B_r(x_0)$ containing Ω so that $u \in W^{1,p}(B_r(x_0))$.

The proof of this inequality relies on the following bound.

Lemma 4.44 (Bound for the Riesz Potential)**.** *For all $p \in [1,n)$ there is an $C = C(n,p)$ such that for all $f \in L^p(\mathbb{R}^n)$ it holds*

$$\|V_1 f\|_{p^*} \le C\|f\|_p$$

where

$$V_\alpha f(x) = \int \frac{f(y)}{\|x-y\|^{n-\alpha}} dy$$

for $\alpha > 0$.

The lemma provides an easy proof of Gagliardo–Nirenberg's Sobolev inequality: For this note that for all $u \in C_c^1(\mathbb{R}^n)$ it holds

$$u(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\sum_{i=1}^n (x_i - y_i)\partial_i u(y)}{\|x-y\|^n} dy.$$

In particular,

$$|u(x)| \le \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\left(\sum_{i=1}^n |\partial_i u|^p(y)\right)^{\frac{1}{p}}}{\|x-y\|^{n-1}} dy.$$

Thus for $f = (\sum_{i=1}^n |\partial_i u|^p(y))^{\frac{1}{p}}$ it holds

$$\|u\|_{p^*} \le \|V_1 f\|_{p^*} \le C\|f\|_p = C\left(\int_{\mathbb{R}^n} \sum_{i=1}^n |\partial_i u|^p(y)dy\right)^{\frac{1}{p}}.$$

We will show that a similar argument also proves the general Sobolev inequality.

Proof of the Sobolev embedding. Note that it suffices to show the inequality for $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Let $\ell = \operatorname{diam}\Omega$ we claim

$$|u(x) - u_\Omega| \le \frac{\ell^n}{n|\Omega|}\int_\Omega \frac{f}{\|x-y\|^{n-1}}dy$$

where $f = (\sum_{i=1}^n |\partial_i u|^p(y))^{\frac{1}{p}}$. Then the claim follows as above.

To see the claim note that

$$u(x) - u(y) = -\int_0^{\|x-y\|} \partial_r u(x + r\omega_{y,x})dr$$

for $\omega_{y,x} = \frac{y-x}{\|y-x\|}$. Thus

$$\begin{aligned}
|u(x) - u_\Omega| &= \frac{1}{|\Omega|}\left|\int_\Omega \int_0^{\|x-y\|} \partial_r u(x + r\omega_{y,x})drdy\right| \\
&\le \frac{1}{|\Omega|}\int_{\|x-y\|<d}\int_0^\infty f(x + r\omega_{y,x})drdy \\
&\le \frac{1}{|\Omega|}\int_0^\infty \int_{\|\omega\|=1}\int_0^d f(x + r\omega_{y,x})\rho^{n-1}d\rho d\omega dr \\
&= \frac{\ell^n}{n|\Omega|}\int_0^\infty \int_{\|\omega\|=1} f(x + r\omega_{y,x})d\omega dr \\
&= \frac{\ell^n}{n|\Omega|}\int_0^\infty \int_{\|\omega\|=1} \frac{f(y)}{\|x-y\|^{n-1}}dy
\end{aligned}$$

where we extended f to a function on $\mathbb{R}^n$ by setting $f = 0$ on $\mathbb{R}^n\backslash\Omega$. □

4.8. **Lax–Milgram Theorem.**

Proposition 4.45 (Riesz respresentation)**.** *Let $(H, \langle\cdot,\cdot\rangle$. Then for every bounded linear map $\alpha : H \to \mathbb{R}$ there is a unique $u \in H$ such that for all $v \in H$ it holds*

$$\alpha(v) = \langle u, v\rangle.$$

Proof. See Exercise Sheet 8. □

Definition 4.46. A bi-linear map $B : X \times X \to \mathbb{R}$is called *bounded* if for some C and all $u, v \in X$ it holds

$$|B(u,v)| \le C \cdot \|u\| \cdot \|v\|$$

it is called *coercive* if for some $c > 0$ and all $u \in X$ it holds

$$B(u,u) \ge C\|u\|^2.$$

Definition 4.47. Let X be a vector space. A symmetric bi-linear map $\langle\cdot,\cdot\rangle$: $X \times X \to \mathbb{R}$ such that $\langle u, u\rangle > 0$ is called a scalar product on X. For $u \in X$ define $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. If $(X, \|\cdot\|)$ is a Banach space then $(X, \langle\cdot,\cdot\rangle)$ will be called a *Hilbert space* and $\|\cdot\|$ its induced norm.

Lemma 4.48. *If $(X, \|\cdot\|)$ admits a bounded, coercive bi-linear map $B : X \times X \to \mathbb{R}$then*

$$\langle u, v\rangle_B = \frac{1}{2}\left(B(u, v) + B(v, u)\right)$$

makes X into a Hilbert space such that $\|\cdot\|$ and the norm $\|\cdot\|_B$ induced by this scalar product are uniformly equivalent.

Theorem 4.49 (Lax-Milgram)**.** *For all bounded bi-linear, coercive map $B : X \times X \to \mathbb{R}$ and all bounded linear maps $\alpha : X \to \mathbb{R}$ there is a unique $u \in X$ such that*

$$B(u, v) = \alpha(v)$$

for all $v \in X$.

Proof. Since B is bounded and coercive there is a scalar product $\langle\cdot,\cdot\rangle$ making X into a Hilbert space such that the norms $\|\cdot\|_B$ and $\|\cdot\|$ are uniformly equivalent. In particular, B is still bounded and coercive with respect to $\|\cdot\|_B$.

By the Riesz Representation Theorem, there is a linear map $T : X \to X$ such that

$$B(u, v) = \langle Tu, v\rangle_B.$$

Note that by the assumptions on B it holds

$$\begin{aligned} c(\|u\|_B)^2 &\le B(u, u) \\ &= \langle Tu, u\rangle_B \\ &\le \|u\|_B \cdot \|Tu\|_B \end{aligned}$$

and

$$\|Tu\|_B^2 \le B(u, Tu) \le C\|u\|_B \cdot \|Tu\|_B.$$

which implies that T is bounded and injective and $T(X)$ is closed. We claim T is also onto. Indeed, if this was false then there is a $z' \in T(X)^\perp\backslash\{\mathbf{0}\}$ such that

$$\langle z', T(u)\rangle_B = 0$$

for all $u \in X$. However, by coercivity, we would have

$$\langle z', T(z')\rangle_B \ge c(\|z'\|_B)^2 > 0$$

which is contradiction.

Let g be the representative of α. Then there is a unique $u = T^{-1}(g)$ such that

$$B(u, v) = \langle g, v\rangle_B = \alpha(v).$$

□

As an application:

Corollary 4.50. *Given a uniformly elliptic operator L on a bounded domain Ω in divergence form. If b^k and c are bounded by a constant only depending on Ω and the lower ellipticity constant λ of L then for all $f \in L^2(\Omega)$ there is a unique $u \in W_0^{1,2}(\Omega)$ such that for all $v \in W_0^{1,2}(\Omega)$ if holds*

$$\int -\sum_{i,j=1}^n a^{ij} \cdot \partial_i u \cdot \partial_j v + \sum_{k=1}^n b^k \cdot \partial_k u \cdot v + c \cdot u \cdot v dx = \int f v dx.$$

In particular, if $u \in W^{2,2}(\Omega)$ then

$$\sum_{i,j=1}^n \partial_j\left(a^{ij} \cdot \partial_i u\right) + \sum_{k=1}^n b^k \cdot \partial_k u + cu = f.$$

4.9. **(to be written) Trace and Extension operators.**

4.10. **(partially covered in exercise) A weak topology via convex sets.**

Definition 4.51 (co-convex topology)**.** Given a Banach space first a *base* for the topology by

$\sigma_{co} = \{U \in 2^X \mid \exists k \in \mathbb{N}, C_1, \dots C_k$ closed and convex such that $U = X \setminus \cup_{i=1}^k C_i\}$.

Then the following defines a topology

$$\tau_{co} = \{U \in 2^X \mid \exists U_i \in \sigma_{co}, i \in I : U = \bigcup_{i \in I} U_i\}.$$

Remark. This can be shorten to saying that τ_{co} is the topology generated by the *subbase* of sets which are *complements* of closed and *convex* subsets. Hence the name *co-convex topology.*

From the definition we see that each set in σ_{co} (hence also each set in τ_{co}) is open with respect to the norm topology. Hence the co-convex topology is weaker than the norm topology which is usually called the strong convergence. Whereas we write $v_n \to v$ for the norm convergence we use $v_n \rightharpoonup v$ for the co-convex topology[9]. In particular, if $v_n \to v$ in X then $v_n \rightharpoonup v$.

Definition 4.52 (Convex hull)**.** Let $A \subset X$ be any subset. Then the closed convex hull of A is defined as

$$\mathsf{conv} A := \mathrm{cl}\{v \in X \mid \exists \lambda_n^m \in [0,1], v_n^m \in A : \sum \lambda_n^m = 1, v = \lim_{m \to \infty} \sum_{n \in \mathbb{N}} \lambda_n^m v_n^m\}.$$

Note that we may equally define by requiring that the convex combinations on the right hand side are finite, i.e. for each m there is an $N_m \in \mathbb{N}$ such that $\lambda_n^m = 0$ whenever $n \geq N_n$.

As a direct corollary of the definition of τ_{co} we obtain the following theorem.

Theorem 4.53 (Mazur's Lemma)**.** *If $(v_n)_{n \in \mathbb{N}}$ converges to v with respect to τ_{co} then*

$$v \in \bigcap_{m \in \mathbb{N}} \mathsf{conv}\{v_n\}_{n \geq m}.$$

In particular, there is sequences $(\lambda_n^m)_{n \in \mathbb{N}}$ in $[0,1]$ and $N_m > m$ in $\mathbb{N}$ with $N_m \to \infty$ such that $\lambda_m^n = 0$ for $n \notin [m, N_m]$, $\sum_{n \in \mathbb{N}} \lambda_n^m = 1$ and

$$v = \lim_{m \to \infty} \sum_{n=1}^{\infty} \lambda_n^m v_n^m.$$

Proof. The subsequence $(v_n)_{n \geq m}$ also converges to v in τ_{co} . Furthermore, the sets $C_m = \mathsf{conv}\{v_n\}_{n \geq m}$ are closed and convex. Since by definition C_m is closed in τ_{co} its τ_{co}-limit v must be in C_m as well. Thus by definition of $\mathsf{conv}\{v_n\}_{n \geq m}$ there are $\lambda_n^m \in [0,1]$ and a $N_m \in \mathbb{N}$ such that $\lambda_n^m = 0$ for $n \notin [m, N_m]$ and

$$\|v - \sum_{n \geq m} \lambda_n^m v_n\| \leq \frac{1}{m}.$$

□

[9]below it will be shown that the co-convex topology is the weak topology.

Lemma 4.54. *If $f : X \to \mathbb{R}$ is a convex lower-semicontinuous function then f is lower semi-continuous with respect to the co-convex topology.*

Proof. The proof is shown for sequences but by change of notation also works for general nets.

Let choose any sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \rightharpoonup v$. Then there is a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that

$$C = \lim_{k\to\infty} f(v_{n_k}) = \liminf_{n\to\infty} f(v_n).$$

Because f is convex and lower semi-continuous the sets $\{f \le r\}$, $r \in \mathbb{R}$, are closed and convex. Thus it suffices to show that $v \in \{f \le C + \epsilon\}$ for all $\epsilon > 0$.

Now for every $\epsilon > 0$ there is an $K \in \mathbb{N}$ such that for all $k \ge K$ it holds $f(v_{n_k}) \le C + \epsilon$ implying that $v_{n_k} \in \{f \le C + \epsilon\}$. However, the set $\{f \le C + \epsilon\}$ is closed and convex which imply that the limit v of $(v_{n_k})_{k\in\mathbb{N}}$ is in $v \in \{f \le C+\epsilon\}$. □

Let X^* be the set of bounded linear functions, i.e. $\alpha : X \to \mathbb{R}$ is linear and there is a C such that

$$|\alpha(v)| \le C\|v\|.$$

On can show that

$$\|\alpha\|^* = \sup_{\|v\|=1} |\alpha(v)|$$

is a norm on X^* making it into a Banach space.

Lemma 4.55. *A linear function $\alpha : X \to \mathbb{R}$ is continuous if and only if it is bounded.*

Proof. Assume first $\alpha \in X^*$. Let $v_n \to v$ then $\|v_n - v\| \to 0$. Thus by linearity and boundedness of α it holds

$$\lim_{n\to\infty} |\alpha(v) - \alpha(v_n)| = \lim_{n\to\infty} |\alpha(v - v_n)| \le C \lim_{n\to\infty} \|v_n - v\| = 0.$$

Assume α is unbounded. We will show that α is not continuous at the origin $\mathbf{0} \in X$. Indeed, unboundedness implies there is a sequence v_n with $\|v_n\| = 1$ such that $0 < a_n := \alpha(v_n) \to \infty$. Observe $w_n := \frac{1}{a_n} v_n \to 0$ so that

$$\lim_{n\to\infty} \alpha(w_n) = 1 \ne \alpha(\mathbf{0}) = 0.$$

□

Definition 4.56 (Weak topology)**.** The weak topology τ_w on X is the weakest/smallest topology on X such that each bounded linear function is continuous.

Corollary 4.57. *Any bounded linear function is continuous with respect to the co-convex topology. In particular, the co-convex topology is stronger*[10] *than the weak topology, i.e. if $(v_n)_{n\in\mathbb{N}}$ converges with respect to τ_{co} then it also converges with respect to τ_w.*

Using the Hahn–Banach Separation Theorem we can also show the following. As we don't need the result in the general setting, we only sketch its proof. The Hahn–Banach Theorem also implies that the τ_w (and thus τ_{co}) are Hausdorff topologies, i.e. a sequence/net can converge to at most one point. We only state the version needed here.

[10]More precisely, "not weaker" as the two topologies are equivalent/the same.

Theorem (Hahn–Banach Separation Theorem)**.** *For any Banach space $(X, \|\cdot\|)$ and every $v \in X\backslash\{\mathbf{0}\}$ and every convex closed set C with $v \notin C$ there is a linear function α with $\alpha(v) \notin [\inf_{w\in C}\alpha(w), \sup_{w\in C}\alpha(w)]$.*

Proof. Without proof. See any textbook on functional analysis. □

Corollary. *For any Banach space it holds $\tau_{co} = \tau_w$, i.e. $v_i \to v$ in τ_{co} if and only if $v_i \to v$ in τ_w for any net $(v_i)_{i\in I}$.*

Proof. By the corollary above it suffices to show that $v_i \to v$ in τ_w implies that $v_i \to v$ in τ_{co}. If C is a convex set then the Hahn–Banach Separation Theorem implies that there is an α such that

$$\alpha(v) \notin [\inf_{w\in C}\alpha(w), \sup_{w\in C}\alpha(w)].$$

In particular, by τ_w-continuity of α there is an $i_0 \in I$ such that

$$\alpha(v_i) \notin [\inf_{w\in C}\alpha(w), \sup_{w\in C}\alpha(w)].$$

Thus $v_i \in X\backslash C$ for $i \geq i_0$. Now let $U \in \sigma_{co}$ with $v \in U$. Then there are closed convex sets C_k and α_k for $k = 1, \ldots, n$ such that

$$U = X\backslash \cup_{k=1}^n C_k$$

and by continuity of α_k there is an $i_0 \in I$ such that

$$\alpha_k(v_i) \notin [\inf_{w\in C_k}\alpha_k(w), \sup_{w\in C_k}\alpha_k(w)] \quad \text{for all } i \geq i_0.$$

But then $v_i \notin C_k$ for all $i \geq i_0$ and $k = 1, \ldots, n$ showing that $v_i \in U$ for $i \geq i_0$.

Finally, let $V \in \tau_{co}$ be an arbitrary neighborhood of v then there is a $U \in \sigma_{co}$ such that $U \subset V$. As above there is an $i_0 \in I$ such that $v_i \in U \subset V$ for all $i \geq i_0$. As V was arbitrary we have shown that $v_i \to v$ in τ_{co}. □

Remark. The proof indicates that it suffices to look at the sets $\{X\backslash C \,|\, C \text{ is closed and convex}\} \subset \tau_w$ which is a subbase for τ_w. A similar argument below will show that a set is compact if and only if every cover formed by elements of this subbase admits a finite subcover, this result − called the Alexander Subbase Theorem − holds more generally for topologies obtained from a subbase.

4.11. **Difference quotients of Sobolev functions.** Given $i \in \{1, \ldots, n\}$, a real number h and a function $u \in L^p(\Omega)$ we define the following

$$\partial_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

It is easy to see that $\partial_i^h u \in L^p(\Omega)$ if $u \in L^p(\Omega)$. In general, however, the L^p-norm of $\partial_i^h u$ is unbounded as can be seen from the following proposition.

Proposition 4.58. *Assume $u \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$ if and only if there is a $K > 0$ such that for all $\Omega' \subset\subset \Omega$, $0 < h < d(\Omega', \partial\Omega)$ and $i \in \{1, \ldots, n\}$ it holds*

$$\|\partial_i^h u\|_{L^p(\Omega')} \leq K.$$

Remark. If $u \in W^{1,p}(\Omega)$ then one can show that $K = \|Du\|_p$ sastisfies the assumption.

Proof. We first show that
$$\|\partial_i^h u\|_p \le \|\partial_i u\|_p.$$
Indeed, if $u \in C^1(\Omega)$ then for fixed $x \in \Omega'$ and $h < d(\Omega', \partial\Omega)$ it holds
$$|\frac{u(x+he_i)-u(x)}{\frac{1}{h}}|^p \le \frac{1}{h}\int_0^h |\partial_i u(x+te_i)|^p dt$$
which yields
$$\begin{aligned}\int_{\Omega'} |\partial_i^h u(x+he_i)|^p dx &\le \int_{\Omega'} \frac{1}{h}\int_0^h |\partial_i u(x+te_i)|^p dtdx \\ &= \frac{1}{h}\int_0^h \int_{\Omega'} |\partial_i u(x+te_i)|^p dxdt \\ &\le \frac{1}{h}\int_0^h \int_{\Omega} |\partial_i u(x+te_i)|^p dxdt \\ &= \int_{\Omega} |\partial_i u(x+te_i)|^p dxdt\end{aligned}$$

Assume now that $\|\partial_i^h u\|_{L^p(\Omega')} \le K$ for all $\Omega' \subset\subset \Omega$. Fix Ω' and observe that the bound implies that $\partial_i^{h_n} u \rightharpoonup g_i$ weakly in $L^p(\Omega')$ for a sequence $h_n \to 0$.

Let $\varphi \in C_c^1(\Omega')$ and $h_n < d(\Omega', \partial\Omega)$ for all large n. Then
$$\int_\Omega \partial_i^{h_n} u \cdot \varphi dx = -\int u \cdot \partial_i^{-h_n}\varphi dx.$$
By the weak convergence we see that the left hand side converges to
$$\int_{\Omega'} g_i \cdot \varphi dx.$$
Since φ has compact support and is differentiable we have $\partial_i^{-h_n}\varphi \to \partial_i\varphi$ uniformly implying that
$$\int_{\Omega'} g_i \cdot \varphi dx = -\int_{\Omega'} u \cdot \partial_i \varphi$$
which shows that g_i is a weak derivative of u restricted to Ω'. Furtermore, g_i is bounded by K. Since weak derivatives are unique on $\Omega_1 \cap \Omega_2$ for two domains $\Omega_1, \Omega_2 \subset\subset \Omega$ we see that g_i can be defined on all of Ω with bound only depending on K. In particular, u has weak derivatives $g_i \in L^p(\Omega)$, $i = 1, \dots, n$, implying that $u \in W^{1,p}(\Omega)$. □

4.12. **(not covered) Maximal Function Theorem.** The following can be derived using the fact that any open set is given by
$$U = \bigcup_{n\in\mathbb{N}, q\in\mathbb{Q}, B_q(x_n)\subset U.} B_q(x_n)$$
where $\{x_n\}_{n\in\mathbb{N}}$ is a fixed dense subset of $\mathbb{R}^n$.

Lemma 4.59. *Given any open cover $\{U_i\}_{i\in I}$ of a subset E of $\mathbb{R}^n$ there is an at most countable $I' \subset I$ such that $\{U_i\}_{i\in I'}$ still covers E.*

In the following given a ball $B = B_r(x)$ for some $x \in \mathbb{R}^n$ and $r > 0$ we denote by kB the ball $B_{kr}(x)$.

Lemma 4.60 (5r-Covering Lemma)**.** *Assume $I \subset \mathbb{N}$ and $\{B_i\}_{i\in I}$ is a collection of balls with* $\sup\{\operatorname{rad} B_i\} < \infty$*. Then there is a subset of indices $J \subset I$ such that the subcollection $\{B_i\}_{i\in J}$ of balls is disjoint and*

$$\bigcup_{i\in I} B_i \subset \bigcup_{i\in J} 5B_i.$$

Exercise (Vitali Covering Lemma for finitely many balls)**.** Show that if I is finite then $J \subset I$ can be chosen such that the balls $\{B_i\}_{i\in J}$ are disjoint and $\bigcup_{i\in I} B_i \subset \bigcup_{i\in J} 3B_i$.

Proof of the lemma. Let $R = \sup\{\operatorname{rad} B_i\}$ and we define

$$I_n = \{i \in I \mid \mathrm{B_i} \in (2^{-(n+1)}R, 2^{-n}].$$

We define inductively sequences J_n and K_n as follows: Let $K_0 = I_0$ and $J_0 \subset K_0$ such that $\{B_i\}_{i\in J_0}$ is disjoint and for any $i \in K_0\backslash J_0$ there is a $i' \in J_0$ with $B_i \cap B_{i'} = \varnothing$, i.e. $\{B_i\}_{i\in J_0}$ is maximal disjoint subcollection of $\{B_i\}_{i\in K_0}$. Assume now K_n and J_n are constructed. Define

$$K_{n+1} = \{i \in I_{n+1} \,|\, B_i \cap B_j = \varnothing \text{ for all } j \in J_0 \cup \ldots \cup J_n\}.$$

As above choose J_{n+1} such that $\{B_i\}_{i\in J_{n+1}}$ is a maximal disjoint subcollection of $\{B_i\}_{i\in K_{n+1}}$.

We claim that $J = \bigcup_{n\in\mathbb{N}} J_n$ sastisfies the assumptions of the claim. Indeed, by construction the collection $\{B_i\}_{i\in\mathbb{N}}$ is disjoint. Furthermore, for $i \in I\backslash J$ there is an $n \in \mathbb{N}$ such that $i \in I_n$. By triangle inequality it suffices to show that there is an $j \in J$ such that $B_i \cap B_j \neq \varnothing$.

Since $i \in I_n$ either $B_i \cap B_j \neq \varnothing$ for some $j \in J_0 \cup \ldots \cup J_{n-1}$ or $i \in K_n$. In the latter case, the choice of J_n implies that the collection

$$\{B_i\} \cup \{B_j\}_{j\in J_n}$$

is not disjoint, i.e. $B_i \cap B_j = \varnothing$ for some $j \in J_n$. □

Definition 4.61 (Hardy–Littlewood maximal function)**.** Given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ and $R \in (0, \infty]$, the (Euclidean) maximal functions $\mathcal{M}_R f$ are defined as

$$\mathcal{M}_R u(x) = \sup_{r\in(0,R)} \fint_{B_r(x)} |u| dx.$$

Remark. If $u : \Omega \to \mathbb{R}$ is measurble we may assume $u = 0$ outside of Ω . This allows us to apply the maximal operator $\mathcal{M}$ also functions defined only on subdomains of $\mathbb{R}^n$.

It is not difficult to see that for $u, v : \mathbb{R}^n \to \mathbb{R}$ measurable and $\lambda \in \mathbb{R}$ it holds

$$\mathcal{M}_R(u + v) \le \mathcal{M}_R u + \mathcal{M}_{\mathcal{R}} v$$

and

$$\mathcal{M}_R(\lambda u) = |\lambda|\mathcal{M}_R u.$$

Theorem 4.62 (Hardy–Littlewood)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then the following statements hold:*

(1) *There is a $C_1 \in (0, \infty)$ such that for all $t > 0$ it holds*

$$\lambda^n(\{\mathcal{M}_R u > t\}) \le \frac{C_1}{t} \int |u| dx.$$

(2) For all $p > 1$ there is a $C_p \in (0, \infty)$ such that

$$\|\mathcal{M}u\|_p \leq C_p \|u\|_p.$$

Proof. Since the bounds are independent of $R \in (0, \infty]$ and $\mathcal{M}_\infty u = \lim_{R\to\infty} \mathcal{M}_R u$ it suffices to assume $R \in (0, \infty)$.

(1): If $u \notin L^1(\mathbb{R}^n)$ then there is nothing to prove. Assume $u \in L^1(\mathbb{R}^n)$.

By definition, for all $x \in \{\mathcal{M}_R u > t\}$ there is an $r_x \in (0, R)$ such that

$$⨍_{B_{r_x}(x)} |u| \geq t.$$

Since $\{B_{r_X}\}_{x\in E_t}$ is a cover of E_t we first pick a countable subcover and then use the $5r$-Covering Lemma to obtain disjoint ball $\{B_i\}_{i\in I}$ such that

$$E_t \subset \bigcup_{i\in I} 5B_i.$$

Then

$$\begin{aligned}
\lambda^n(\{\mathcal{M}_R u > t\}) &\leq \sum_{i\in I} \lambda^n(5B_i) \\
&\leq 5^n \sum_{i\in I} \lambda^n(B_i) \\
&\leq \frac{5^n}{t} \sum_{i\in I} \int_{B_i} |u| dx \leq \frac{5^n}{t} \int |u| dx.
\end{aligned}$$

(2): Again if $u \notin L^p(\mathbb{R}^n)$ then there is nothing to prove. So assume $u \in L^p(\mathbb{R}^n)$. Furthermore, since $\mathcal{M}u = \mathcal{M}|u|$ we may assume $u \geq 0$.

For $t > 0$ note that

$$u \leq (u - \frac{t}{2})_+ + \frac{t}{2}$$

where $(u - \frac{t}{2})_+ = \max\{u - \frac{t}{2}, 0\}$. Then

$$\{\mathcal{M}_R u > t\} \subset \{\mathcal{M}_R (u - \frac{t}{2})_+ > \frac{t}{2}\}.$$

Using Cavalleri's Principle, the bound obtained for $p = 1$ appied to $(u - \frac{t}{2})$we obtain

$$\begin{aligned}
\int |\mathcal{M}_R u|^p d\mu &= p \int_0^\infty t^{p-1} \lambda^n(\{\mathcal{M}_r u > t\}) dt \\
&\leq p \int_0^\infty t^{p-1} \lambda^n(\{\mathcal{M}_r (u - \frac{t}{2})_+ > \frac{t}{2}\}) dt \\
&= p 2^p \int_0^\infty t^{p-1} \lambda^n(\{\mathcal{M}_r (u - t)_+ > t\}) dt \\
&\leq C_1 p 2^p \int_0^\infty t^{p-2} \int_{\mathbb{R}^n} u \chi_{\{u > t\}} dx dt \\
&= C_1 p 2^p \int_{\mathbb{R}^n} \int_0^\infty t^{p-2} u \chi_{\{u > t\}} dt dx \\
&= C_1 p 2^p \int_{\mathbb{R}^n} u(x) \int_0^{u(x)} t^{p-2} dt dx \\
&= C_1 \frac{p 2^p}{p-1} \int u^p dx.
\end{aligned}$$

so that the result holds for $C_p = \left(C_1 \frac{p 2^p}{p-1} \right)^{\frac{1}{p}}$. □

5. L^2-Estmates

In the following let L be a uniformly elliptic operator such that $a^{ij}, b^k, c : \bar{\Omega} \to \mathbb{R}$ are Lipschitz continuous. Furthermore, let L_0 with $a_0^{ij} = a^{ij}$ and $b_0^k = c_0 = 0$. Note that in this case if $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = \alpha$ for a bounded linear map on $W_0^{1,2}(\Omega)$, i.e. it satisfies

$$B_L(u,v) = \int \sum_{i,j=1}^n a^{ij} \cdot \partial_i u \cdot \partial_i v + \sum_{k=1}^n b^k \cdot \partial_k u \cdot v + c \cdot u \cdot v dx = \alpha(v) \qquad \forall v \in W_0^{1,2}(\Omega)$$

then u is also a weak solution to $L_0 u = \alpha_u$ where

$$\begin{aligned}
\alpha_u(v) &= \alpha(v) - \int \sum_{k=1}^n b^k \cdot \partial_k u \cdot v + c \cdot u \cdot v dx \\
&= \alpha(v) + \int \sum_{k=1}^n (b^k \cdot u \cdot \partial_k v + \partial_k b^k \cdot u \cdot v) - c \cdot u \cdot v dx
\end{aligned}$$

Note that

$$|\alpha_u(v)| \leq |\alpha(v)| + (\sum_{k=1}^n \|b^k\|_{C^{0,1}} + \|c\|_k) \cdot \|u\|_2 \cdot \|v\|_{W^{1,2}}$$

which implies that α_u is also a bounded linear map on $W_0^{1,2}(\Omega)$ with norm depending only on $\|u\|_2$ and the bounds on b^k and c. Note that a natural norm for bounded linear maps $\alpha : W_0^{1,2}(\Omega)$ is given by

$$\|\alpha\| = \sup_{v \in W_0^{1,2}(\Omega) \setminus \{\mathbf{0}\}} |\alpha(v)|.$$

The argument show that it suffices to look at weak solutions $L_0 u = \alpha$.

Proposition 5.1 (Cacciopoly inequality)**.** *Let $u \in W^{1,2}(\Omega)$ be $L_0 u = \alpha$ weakly, i.e.*

$$-\int \sum_{i,j=1}^{n} a^{ij} \cdot \partial_i u \cdot \partial_j v dx = \alpha(v)$$

for all $v \in W_0^{1,2}(\Omega)$. Then for all $\Omega' \subset\subset \Omega$ it holds

$$\|u\|_{W^{1,2}(\Omega')} \leq C(\Omega, \Omega', \lambda, \Lambda, n) \left(\|\alpha\| + \|u\|_{L^2(\Omega)}\right)$$

where

$$\|\alpha\| = \sup_{v \in W_0^{1,2}(\Omega)\setminus\{\mathbf{0}\}} \frac{|\alpha(v)|}{\|v\|_{W^{1,2}}}.$$

Proof. Let $\eta : \Omega \to [0,1]$ be a smooth map with compact support in Ω such that $\eta \equiv 1$ on Ω'. It is not difficult to see that the derivatives of η are bounded by a constant C only depending on the distance of Ω' and $\partial\Omega$.

Now it is easy to see that $v = \eta^2 u$ is a function in $W_0^{1,2}(\Omega)$ so that

$$B_{L_0}(u, v) = \alpha(v).$$

From Gagliardo–Nirenberg we obtain

$$\begin{aligned}
\|v\|_{W^{1,2}} &\leq C(\Omega, n) \cdot \|D(\eta^2 u)\|_2 \\
&\leq C(\Omega, n) \cdot \left(\|\eta^2 |Du|\|_2 + \|\eta\|_{C^1} \|u\|_2\right) \\
&\leq C(\Omega, \Omega', n) \|\eta |Du|\|_2.
\end{aligned}$$

In particular,

$$|\alpha(v)| \leq \|\alpha\| \cdot \|v\| \leq \frac{1}{4\epsilon} \|\alpha\|^2 + \epsilon \int \eta^2 |Du|^2 dx.$$

Since $\eta \equiv 1$ on Ω' and L_0 is uniformly elliptic we obtain

$$\begin{aligned}
\lambda \int_\Omega \eta^2 \cdot |Du|^2 dx &\leq \int_\Omega \sum_{i,j=1}^{n} \eta^2 \cdot a^{ij} \cdot \partial_i u \cdot \partial_j u dx \\
&= \int_\Omega \sum_{i,j=1}^{n} a^{ij} \cdot \partial_i u \cdot \partial_j v dx - \int \sum_{i,j=1} a^{ij} \cdot \partial_i u \cdot \partial_j \eta \cdot 2\eta \cdot u dx \\
&= \alpha(v) + \Lambda \int \eta \cdot |Du| \cdot |D\eta| \cdot |u| dx \\
&\leq \|\alpha\| \|v\|_{W^{1,2}} + \Lambda \|\eta \cdot |Du|\|_2 \cdot \|2 \cdot |D\eta| \cdot |u|\|_2 \\
&\leq \frac{1}{4\epsilon} (\|\alpha\|^2 + \Lambda \|\eta\|_{C^1} \|u\|_2^2) + \epsilon(1+\Lambda) \int \eta^2 \cdot |Du|^2 dx.
\end{aligned}$$

Now choosing ϵ small (only depending on λ and Λ) such that $\epsilon(1+\Lambda) < \frac{\lambda}{2}$ we obtain

$$\frac{\lambda}{2} \int \eta^2 \cdot |Du|^2 \leq \frac{1}{4\epsilon} (\|\alpha\|^2 + \Lambda \|\eta\|_{C^1} \|u\|_2^2).$$

Since ϵ and $\|\eta\|_{C^1}$ only depend on the data and $\eta \equiv 1$ on Ω' we obtain

$$\int_{\Omega'} u^2 dx + \int_{\Omega'} |Du|^2 dx \leq C(\Omega, \Omega', \lambda, \Lambda, n) \left(\|\alpha\|^2 + \|u\|_{L^2(\Omega)}^2\right).$$

□

Theorem 5.2 (L^2-estimate)**.** *Assume u is a weak solution to $L_0u = f$ in Ω. If a^{ij} are Lipschitz functions in $\bar{\Omega}$ then $u \in W^{2,2}_{loc}(\Omega)$ and for all $\Omega' \subset\subset \Omega$ it holds*

$$\|u\|_{W^{2,2}(\Omega')} \leq C(\Omega, \Omega', \lambda, \Lambda, n)(\|\alpha\|_{W_0^{1,2}(\Omega)^*} + (1 + \|a^{ij}\|_{C^{0,1}}) \cdot \|u\|_{L^2(\Omega)}).$$

In particular, if $Lu = f$ and b^k is also Lipschitz on $\bar{\Omega}$ then

$$-\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + \sum_{k=1}^n b^k \partial_k u + cu = f \quad \text{almost everywhere in } \Omega.$$

Proof. Note that the last result follows by observing that $u \in W^{2,2}_{loc}(\Omega)$ and

$$B_L(u, \varphi) = \int f\varphi dx$$

for all $\varphi \in C_c^\infty(\Omega)$ implies that we can apply partial integration to the quadratic term in B_L which yields

$$\int \left(-\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + \sum_{k=1}^n b^k \partial_k u + cu \right) \cdot \varphi dx = \int f\varphi dx.$$

Back to the equation $L_0u = \alpha$. Let $\Omega' \subset\subset \Omega$ and choosing in the following $0 < h < d(\Omega', \partial\Omega)$. Then for all $v \in W_0^{1,2}(\Omega')$ it holds $v(x \pm he_k) \in W_0^{1,2}(\Omega)$.

Note that for $v \in W_0^{1,2}(\Omega')$ it holds

$$\begin{aligned}\int \sum_{i,j=1}^n a^{ij}\partial_i(\partial_k^h u)\partial_j v dx &= -\int \sum_{i,j=1}^n \partial_i u \partial_k^{-h}\left(a^{ij} \cdot \partial_j v\right) dx \\ &= -\int \sum_{i,j=1}^n a^{ij}\partial_i u \partial_j(\partial_k^{-h} v) dx - \int \sum_{i,j=1}^n (\partial_k^{-h} a^{ij})\partial_i u \cdot \partial_j v(\cdot - he_k) dx \\ &= \int f(\partial_k^{-h} v) dx - \int \sum_{i,j=1}^n (\partial_k^h a^{ij})\partial_i u(\cdot + he_k) \cdot \partial_j v dx =: \alpha^h(v),\end{aligned}$$

i.e. $L_0\partial_k^h u = -\alpha^h$ weakly.

Let now Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and $0 < h < d(\Omega', \partial\Omega'')$ then from the Cacciopoli inequality we obtain

$$\begin{aligned}\alpha^h(v) &\leq \|f\|_2\|\partial_k^{-h} v\|_{L^2(\Omega'')} + \|a^{ij}\|_{C^{0,1}(\Omega'')} \cdot \|u\|_{W^{1,2}(\Omega'')} \cdot \|v\|_{W_0^{1,2}(\Omega)} \\ &\leq (\|f\|_2 + \|a^{ij}\|_{C^{0,1}(\Omega'')} \cdot \|u\|_{L^2(\Omega)}) \cdot \|v\|_{W_0^{1,2}(\Omega)}.\end{aligned}$$

Thus Cacciopoli gives

$$\|\partial_k^h u\|_{W^{1,2}(\Omega')} \leq C(\Omega, \Omega'', \Omega', \Lambda, \lambda, n)\left(\|f\|_2 + (1 + \|a^{ij}\|_{C^{0,1}(\Omega'')}) \cdot \|u\|_{L^2(\Omega)}\right).$$

Finally, we may choose Ω'' such that the constant does not depend on Ω'' anymore. □

Corollary 5.3 (Higher L^2-regularity)**.** *Assume $f \in W^{k,2}(\Omega)$ and $Lu = f$ weakly. If $a^{ij}, b^k \in C^{k-1,1}(\Omega)$ and $c \in C^{k-1}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$ and for all Ω' there is a constant C depending the data such that*

$$\|u\|_{W^{k+2,2}(\Omega')} \leq C(\|f\|_{W^{k,2}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Proof. We only give the idea for the $W^{3,2}$-regularity: From the proof above we have

$$\int \sum_{i,j=1}^n a^{ij}\partial_i(\partial_k^h u)\partial_j v dx = -\int \partial_k^h f v dx - \int \sum_{i,j=1}^n (\partial_k^h a^{ij})\partial_i u(\cdot + he_k)\cdot \partial_j v dx$$

so that

$$\int \sum_{i,j=1}^n a^{ij}\partial_i(\partial_k u)\partial_j v dx = -\int \partial_k^h f v dx - \int \sum_{i,j=1}^n (\partial_k a^{ij}\partial_i u)\cdot \partial_j v dx$$
$$= -\int (\partial_k f - \sum_{i,j=1}^n \partial_j(\partial_k a^{ij}\partial_i u))\cdot v dx = \int \hat{f}_i \cdot v dx.$$

If $v \in W_0^{1,2}(\Omega')$ for $\Omega' \subset\subset \Omega$ then

$$\|\hat{f}\|_2 \le \sum_{i,j=1}^n \|a^{ij}\|_{C^{1,1}}\|u\|_{W^{2,2}}.$$

This shows that $\partial_k u \in W^{2,2}_{loc}(\Omega)$. Using nested domains we can therefore bound the $W^{3,2}$-norm of u on $\Omega' \subset \Omega$ by the $C^{1,1}$-norm of a^{ij}. and the L^2-norms of f and u as well. □

Corollary 5.4 (Inner C^∞-regularity)**.** *Assume $f \in C^\infty$ and $Lu = f$ weakly. If $a^{ij}, b^k, c \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

Proof. Just observe that $u \in W^{k,2}_{loc}(\Omega)$ for all $k \in \mathbb{N}$. Thus using a smooth cut-off function η which is constant equal to one on $\Omega' \subset\subset \Omega$ we see $\eta u \in W_0^{k,2}(\Omega)$ for all $k \in \mathbb{N}$ so that via Sobolev and Morrey embedding we obtain $\eta u \in C^l(\Omega)$ for all $l \in \mathbb{N}$. Since $u = \eta u$ on Ω', we see that u is differentiable of all orders in Ω' (more precisely u agrees with an infinitely differentiable function on Ω'). □

Lemma 5.5 (boundary Cacciopoli inequality)**.** *Let $\Omega = B_2(\mathbf{0})^+ = \{x \in B_2(\mathbf{0}) \,|\, x_n \ge 0\}$ and L_0 be as above and Then for any bounded linear map $\alpha : W_0^{1,2}(B_2(\mathbf{0})^+) \to \mathbb{R}$ and $\varphi \in W^{1,2}(B_2(\mathbf{0})^+)$. Then there is a constant $C = C(\lambda, \Lambda, n)$ such that for any weak solution $L_0 u = \alpha$ with $u - \varphi \in W_0^{1,2}(B_2(\mathbf{0})^+)$ it holds*

$$\|u\|_{W^{1,2}(B_1(\mathbf{0})^+)} \le C(\|\alpha\| + \|\varphi\|_{W^{1,2}(B_2(\mathbf{0})^+)} + \|u\|_{L^2(B_2(\mathbf{0})^+)}).$$

Proof. Note that because

$$\|u\|_{W^{1,2}(B_1(\mathbf{0})^+)} \le \|u-\varphi\|_{W^{1,2}(B_1(\mathbf{0})^+} + \|\varphi\|_{W^{1,2}(B_2(\mathbf{0})^+)}$$

and

$$B_L(u-\varphi, v) = \alpha(v) - B_L(\varphi, v) =: \alpha_\varphi(v)$$

where $\|\alpha\|_\varphi \le \|\alpha\| + \|\varphi\|_{W^{1,2}(B_2(\mathbf{0})^+)}$, it suffices to show the estimate for $\varphi \equiv 0$.

In this case observe that for a smooth cut-off function $\eta : B_2(\mathbf{0}) \to [0,1]$ with $\eta \equiv 1$ on $B_1(\mathbf{0})$ the function $v := \eta^2 u$ is in $W_0^{1,2}(B_2(\mathbf{0})^+)$. Thus the same argument as in proof of the Cacciopoli inequality applies. □

Theorem 5.6 (Global regularity)**.** *Assume Ω is a bounded open domain with $C^{k+1,1}$-boundary. If L is uniformly elliptic and $a^{ij}, b^k \in W^{k+1,\infty}(\Omega)$ and $c \in W^{k-1,\infty}(\Omega)$ then for any $f \in W^{k,2}(\Omega)$ and $\varphi \in W^{k+2,2}(\bar{\Omega})$ there is an $u \in$*

$W^{k+2,2}(\Omega)$ solving the PDE $Lu = f$ (weakly) in Ω and having trace $Tu \in W^{k,2}(\partial\Omega)$ equal to $T\varphi$. Furthermore, it holds

$$\|u\|_{W^{k+2,2}} \leq C(\Omega, \lambda, \Lambda, n)(\|f\|_{W^{k,2}} + \|\varphi\|_{W^{k+2,2}} + \|u\|_{L^2}).$$

Sketch of the Proof. If $\Omega = B_2(\mathbf{0})$ the proof follows exactly as the interior regularity proof by using the boundary Cacciopoli inequality. For the general case note that locally near every $x_0 \in \partial\Omega$ we can use $C^{1,1}$-diffeomorphism $\Psi : U^+_{x_0} \to B_2(\mathbf{0})^+$ such that $\tilde{u} = u \circ \Psi$ solves a uniformly elliptic PDE. More precisely, there is a uniformly elliptic operator $\tilde{L}$ with

$$\tilde{a}^{kl} = \sum_{i,j=1}^{n} (a^{ij}\partial_i\Psi^l\partial_j\Psi^k) \circ \Psi^{-1} \cdot |\det(D\Psi^{-1})|$$
$$\tilde{b}^p = \sum_{k=1}^{n} (b_k\partial_k\Psi^p) \circ \Psi^{-1} \cdot |\det(D\Psi^{-1})|$$

and

$$\tilde{c} = c \circ \Psi^{-1} \cdot |\det(D\Psi^{-1})|$$
$$\tilde{f} = f \circ \Psi^{-1} \cdot |\det(D\Psi^{-1})|$$

and it holds $\tilde{L}\tilde{u} = \tilde{f}$ weakly.

Since $\Psi \in C^{1,1}$ and $a^{ij}, b^k \in C^{0,1}$ it holds $\tilde{a}^{kl}, \tilde{b}^p \in C^{0,1}$ with norms depending only on Ψ. Similarly, the uniform ellipticity constants of $\tilde{L}$ only depend on Ψ and the constants of L.

Now for $k = 1, \ldots, n-1$ we can apply the argument of the inner regularity to conclude that $\partial_k\tilde{u} \in W^{1,2}(B_1(\mathbf{0}))$ with corresponding bounds depending on the L^2-bounds of $\tilde{u}$ and $\tilde{f}$.

For $k = n$ observe that

$$\partial_{nn}\tilde{u} = (\tilde{a}^{nn})^{-1}\left(-\sum_{(k,l)\neq(n,n)} \tilde{a}^{kl}\partial_{kl}\tilde{u} - \sum_{k,l=1}^{n}(\partial_k\tilde{a}^{kl})\partial_l\tilde{u} + \tilde{f}\right) \quad \text{in } B_2(\mathbf{0})^+.$$

Since $(\tilde{a}^{nn})^{-1}$ is bounded by a constant depending only on L and Ψ, we see that the Cacciopoli inequality gives

$$\|\partial_{nn}\tilde{u}\|_{L^2(B_1(\mathbf{0})^+)} \leq C(\Psi, \lambda, \Lambda)(\|\tilde{f}\|_{L^2(B_2(\mathbf{0})^+)} + (1 + \|\tilde{a}^{kl}\|_{C^{0,1}})\|u\|_{L^2(B_1(\mathbf{0})^+)})$$

which gives the desired $W^{2,2}$-bound of $\tilde{u}$ on $B_1(\mathbf{0})^+$.

Using the uniformy bounds on the derivatives of Ψ we then obtain $W^{2,2}$-bounds for u on U_{x_0}.

Since $\bar{\Omega}$ is compact we may find finitely many U_i so that the $W^{2,2}$-norm is bounded which gives the desired global $W^{2,2}$-bound. □

6. Properties of weak (sub)solution

Definition 6.1. Let L be a uniformly elliptic operator on a bounded domain Ω with $b^k = c = 0$ then $u \in W^{1,2}(\Omega)$ is called a weak subsolution of the equation $Lu = f$ (short $Lu \geq f$ weakly) if for all $v \in W^{1,2}_0(\Omega)$ it holds

$$B_L(u, v) := \int \sum_{i,j=1}^{n} a^{ij} \cdot \partial_i u \cdot \partial_j u dx \leq -\int f \cdot v dx.$$

Note that if $u \in W^{1,2}(\Omega)$ then also $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and $|u| = u^+ + u^-$ are in $W^{1,2}(\Omega)$. Now if Ω has a Lipschitz boundary then the trace $T : W^{1,2}(\Omega) \to L^2(\Omega)$ is well-defined and $u \in W_0^{1,2}(\Omega)$ if and only if $Tu = 0$. Thus by linearity we have

$$Tu = T(u^+) - T(u^-)$$

and $Tu^\pm \geq 0$ which implies

$$\operatorname{ess\,sup}_{\partial\Omega} Tu = \|T(u^+)\|_{L^\infty(\partial\Omega)} = \inf\{M \in \mathbb{R} \,|\, (u - M)^+ \in W_0^{1,2}(\Omega)\}.$$

Note finally that the right hand expression makes sense in general.

Proposition 6.2 (Weak maximum principle)**.** *If $Lu \geq 0$ weakly for $u \in C^0(\Omega) \cap W^{1,2}(\Omega)$ and $(u - M)^+ \in W_0^{1,2}(\Omega)$ then $u \leq M$ in Ω.*

Proof. The assumptions imply $v = (u - M)^+$ can be used as test function. Thus

$$\int_{\{u>M\}} \sum_{i,j=1}^n a^{ij} \cdot \partial_i u \cdot \partial_j u dx = \int \sum_{i,j=1}^n a^{ij} \cdot \partial_i u \cdot \partial_j v dx \leq 0.$$

Since L is uniformly elliptic we see that $\partial_i u = 0$ on $\{u > M\}$. This, however implies that u is locally constant in the open set $\{u > M\}$. If $\{u > M\}$ were non-empty this would give a contradiction as Ω is connected. □

Remark. The statement also holds for $u \in W^{1,2}(\Omega)$ with $u \leq M$ a.e. in Ω. For this observe if $\partial_i v = \partial_i u$ almost everywhere on $\{v = u\}$ whenever $u, v \in W^{1,2}(\Omega)$.

Theorem 6.3 (Strong Maximum Principle)**.** *If $Lu \geq 0$ weakly for $u \in C^0(\Omega) \cap W^{1,2}(\Omega)$ and $(u - M)^+ \in W_0^{1,2}(\Omega)$ then either $u \equiv M$ or $u < M$ in Ω.*

Proof. Assume $C = \{x \in \Omega \,|\, u(x) = M\}$ is non-empty and not equal to Ω. Then there is a $x_0 \in \Omega$ and $y \in C$ such that

$$d(x_0, y) = \inf_{y' \in C} d(x, y').$$

Furthermore, let $x_1 = \frac{x_0+y}{2}$ then $B_r(x) \subset \{u < M\}$ and $\partial B_r(x) \cap C = \{y\}$.

As in Lemma 3.7, choose $\alpha \gg 1$ such that

$$Lv(x) \geq 0$$

for $v(x) = e^{-\alpha\|x-y\|^2} - e^{-\alpha r^2}$.

Choose $s \ll 1$ such that $\Omega' := B_s(y) \subset \Omega$ and observe that $v < 0$ on $\partial\Omega' \backslash \bar{B}_r(x_1)$ and $u < M = u(y)$ on $\partial\Omega' \cap \bar{B}_r(x_1)$. Now choose $\epsilon > 0$ such that

$$\epsilon < \frac{M - \inf_{\partial\Omega' \cap \bar{B}_r(x_1)} u}{\sup v}$$

then

$$u_\epsilon := u + \epsilon v\big|_{\partial\Omega'} < M = u(y) + \epsilon v(y).$$

In particular, there is an $\tilde{M} < M$ such that $(u_\epsilon - \tilde{M})^+ \in W_0^{1,2}(\Omega')$. However, the weak maximum principle implies

$$u_\epsilon \leq \tilde{M}$$

which contradicts the fact that $y \in \Omega'$ and the $u_\epsilon(y) = M > \tilde{M}$. □

We also obtain the following result. The proof is similar to the one below using the weak maximum principle.

Corollary 6.4 (A priori bounds)**.** *Let L be a uniformly elliptic operator with $b^k = c = 0$ on a bounded domain Ω such that $\beta := \sup \frac{|b^k|}{\lambda} < \infty$ and $M \geq 0$. Then for all $u \in W^{1,2}(\Omega) \cap C^0(\Omega)$ with $Lu \geq f$ weakly and $(u-M)^+ \in W_0^{1,2}(\Omega)$ it holds*

$$\sup_\Omega u \leq M + \frac{C}{\lambda} \sup_\Omega |f^-|$$

where $C = e^{(\beta+1)\operatorname{diam}\Omega} - 1$.

Proposition 6.5 (Limits of subsolutions)**.** *Assume $(u_n)_{n\in\mathbb{N}}$ is a sequence in $W^{1,2}(\Omega)$ converging weakly to $u \in W^{1,2}(\Omega)$. If $Lu_n \geq f$ weakly then $Lu \geq f$ weakly.*

Proof. Note that $u_n \rightharpoonup u$ weakly implies that for all bounded linear functionals $\alpha : W^{1,2}(\Omega) \to \mathbb{R}$ it holds

$$\lim_{n\to\infty} \alpha(u_n) = \alpha(u).$$

Observe that uniform ellipticity of L implies

$$u \mapsto B_L(u,v)$$

is a bounded linear functional for every fixed $v \in W_0^{1,2}(\Omega)$. In parrticular, we have

$$B_L(u,v) = \lim_{n\to\infty} B_L(u_n,v) \geq \lim_{n\to\infty} -\int f \cdot v dx = -\int f \cdot v dx$$

proving that $Lu \geq f$ weakly. □

7. Schauder esimates

The Sobolev regularity theory shows that provided the coefficients behave nicely and f is sufficiently many times weakly differentiable (and thus Hölder continuous) then $Lu = f$ implies u is a classical solution with Hölder continuous second derivatives. Via so-called L^p-estimates it is possible to obtain $W^{2,p}$-bounds which would yield $C^{1,\alpha}$-solutions provided f is a bounded function.

More classical it is possible to directly obtain $C^{2,\alpha}$-bounds if f is a C^α-function. Those estimates are called Schauder estimates. In order to simplify the notation we define the following semi-norm for $\alpha \in (0,1]$ and $u : \Omega \to \mathbb{R}$ (we omit the domain if it is clear from the context)

$$[u]_{\alpha,\Omega} = \sup_{x\neq y\in\Omega} \frac{|u(x)-u(y)|}{d(x,y)^\alpha}$$

and

$$\|u\|_{k,\alpha,\Omega} = \sum_{|I|\leq k} \|\partial_I u\|_0 + \sum_{|I|=k} [\partial_I u]^*_{\alpha,\Omega}$$

Note that it is easy to see that $[u]^*_{\alpha,\Omega} = 0$ implies $u \equiv \mathsf{const}$.

If $u \in C^{k,\alpha}(\Omega)$ then we use the short hand notation

$$[D^k u]_{\alpha,\Omega} = \sum_{|I|=k} [\partial_I u]^*_{\alpha,\Omega}.$$

Lemma 7.1 (Ehrling's Lemma)**.** *Assume X, Y, Z are three Banach spaces and there are two bounded linear maps $K : X \to Y$ and $I : Y \to Z$ such that K is compact and I is injective. Then for all $\epsilon > 0$ there is a $C = C(\epsilon) > 0$ such that*

$$\|Kx\|_Y \leq \epsilon \|x\|_X + C \cdot \|I(Kx)\|_Z.$$

Proof. If the claim was wrong then fixed $\epsilon > 0$ there is a sequence $(x_m)_{m\in\mathbb{N}}$ in X such that

$$\epsilon\|x_m\|_X + m\cdot\|I(Kx_m)\|_Z \le \|Kx_m\|_Y = 1.$$

Since K compact there is a subsequence such that $Kx_{m_l} \to y$. Note that $1 = \lim_{n\to\infty}\|Kx_{m_l}\| = \|y\|$ which implies $y \neq 0$. By continuity it holds

$$\|Iy\| \le \lim_{m\to\infty}\|I(Kx_m)\| \le \lim_{m\to\infty}\frac{1}{m} = 0.$$

Then injectivity of I shows $y = 0$ which is a contradiction. □

By Arzela–Ascoli we obtain the following.

Corollary 7.2. *For all bounded open sets Ω and all $\epsilon > 0$ there is a constant $C = C(\epsilon) > 0$ such*

$$\|u\|_{C^2} \le \epsilon\|u\|_{C^{2,\alpha}} + C(\epsilon)\|u\|_{C^0}$$

and

$$\|u\|_{C^2} \le \epsilon\|u\|_{C^{2,\alpha}} + C(\epsilon)\|u\|_{L^2}.$$

Lemma 7.3. *Assume u is a functions satisfying the following: for all $\epsilon > 0$ is an $R_\epsilon > 0$ such that*

$$|u(x)| \le \epsilon\|x\|$$

whenever $x \notin B_{R_\epsilon}(\mathbf{0})$ (in short $|u(x)| = o(\|x\|)$ as $\|x\| \to \infty$). If u is harmonic then u is constant.

Proof. Let $x, y \in \mathbb{R}^n$ then

$$\begin{aligned}|B_R(x)\Delta B_R(y)| &= |B_R(x)\backslash B_R(y) \cup B_R(y)\backslash B_R(x)| \\ &\le d(x,y)|\partial B_R(\mathbf{0}) = C_n d(x,y)R^{n-1}.\end{aligned}$$

Let $\epsilon > 0$ and $R > (R_\epsilon + \|x\| + \|y\|)$. Then $B_R(x)\Delta B_R(y) \subset \mathbb{R}^n\backslash B_{R_\epsilon}$ and the mean-value property shows

$$\begin{aligned}|u(x) - u(x')| &\le \frac{1}{|B_R(\mathbf{0})|}\int_{B_R(x)\Delta B_R(y)}|u(z)|dz \\ &\le 2\tilde{C}_n d(x,y)\epsilon.\end{aligned}$$

As ϵ is arbitrary we see that $u(x) = u(y)$ which implies the claim. □

Corollary 7.4. *Assume u is harmonic on $\mathbb{R}^n$ and $[D^k u]_\alpha < \infty$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Then u is a polynomial of order at most k, i.e. $D^k u \equiv \mathsf{const}$.*

Proof. Since $\partial_I u$ is harmonic if u is harmonic, it suffices to show that a harmonic function with $[u]_\alpha \le C < \infty$ is constant. Now the bound implies

$$\begin{aligned}|u(x)| &\le |u(x) - u(\mathbf{0})| + |u(\mathbf{0})| \\ &\le C(1 + \|x\|^\alpha)\end{aligned}$$

which shows that u satisfies the conditions of the previous lemma. In particular, u must be constant. □

Proposition 7.5. *For all $u \in C^{2,\alpha}(\mathbb{R}^n)$ then it holds*

$$[D^2u]_{\alpha,\Omega} \le C(n,\alpha)[\Delta u]_{\alpha,\Omega}.$$

Proof. If the claim was wrong then there is a sequence $(u_m)_{m\in\mathbb{N}}$ in $C^{2,\alpha}_{loc}(\mathbb{R}^n)$ such that
$$m[\Delta u_m]_{\alpha,\Omega} \leq [D^2 u_m]_{\alpha,\Omega} < \infty.$$
Via rescaling and shifting we can assume $[D^2 u_m]_{\alpha,\Omega} = 1$ and
$$\sup_{|I|=2} \sup_{y\in\mathbb{R}^n} \frac{|\partial_I u_m(y) - \partial_I u_m(\mathbf{0})|}{\|y\|^\alpha} \geq \frac{1}{2}.$$
Furthermore, extracting a subsequence $(u_{m_l})_{l\in\mathbb{N}}$ we find a multi-index I with $|I| = 2$, a direction e_i and sequence $(h_l)_{l\in\mathbb{N}}$ such that
$$\frac{|\partial_I u_{m_l}(h_l \cdot e_i) - \partial_I u_{m_l}(\mathbf{0})|}{h_l^\alpha} \geq \frac{1}{4n^2} > 0.$$
Set
$$\tilde{u}_l(x) = h_l^{2-\alpha} u_{m_l}(h_l \cdot x)$$
then still
$$m[\Delta \tilde{u}_l]_{\alpha,\Omega} \leq [D^2 \tilde{u}_m]_{\alpha,\Omega} < \infty$$
and
$$[D^2 u_m]_{\alpha,\Omega} = 1.$$
Furthermore, $|\partial_I \tilde{u}_l(e_i) - \partial_I \tilde{u}_l(\mathbf{0})| \geq \frac{1}{4n^2}$. In addition, we can find a quadratic polynomial $v_l = \langle x, b + Ax\rangle + c$ such that
$$\hat{u}_l(\mathbf{0}) = 0$$
$$D\hat{u}_l(\mathbf{0}) = 0$$
$$D^2\hat{u}_l(\mathbf{0}) = 0$$
and choosing v_l appropriately we also have
$$\hat{u}(e_i) \neq \hat{u}(\mathbf{0})$$
where $\hat{u}_l = \tilde{u}_l - v_l$. Note that $[D^2\hat{u}_l]_\alpha = [D^2\tilde{u}_l]_\alpha = 1$ so that we can extract a subsequence $(\hat{u}_{l_k})_{k\in\mathbb{N}}$ converging locally uniformly in $C^2(\mathbb{R}^n)$ to a function u satisfying
$$\Delta u = \mathsf{const}$$
$$[D^2 u]_\alpha \leq 1$$
$$\partial_I u(e_i) \neq \partial_I u(\mathbf{0})$$
$$Du(\mathbf{0}) = 0$$
$$D^2 u(\mathbf{0}) = 0.$$
Thus u is harmonic and previous result $D^2 u \equiv \mathsf{const}$. However, this means that u would be constant, contradicting the fact that $\partial_I u(e_i) \neq \partial_I u(\mathbf{0})$. □

Corollary 7.6. *If A is a symmetric positive matrix with $\lambda I_n \leq A \leq \Lambda I_n$ then*
$$[D^2 u]_\alpha \leq C(n, \alpha, \lambda, \Lambda)[L_A u]_\alpha.$$
where L_A is the elliptic operator in non-divergence form obtained from A.

Proposition 7.7. *Let L be an uniformly elliptic opertor in non-divergence form such that $a^{ij} \in C^\alpha(B_2(\mathbf{0}))$ then for all $u \in C^{2,\alpha}(B_2(\mathbf{0}))$ it holds*
$$\|D^2 u\|_{C^{2,\alpha}(B_1(\mathbf{0}))} \leq C(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)(\|Lu\|_{C^{0,\alpha}(B_2(\mathbf{0}))} + \|u\|_{C^2(B_2(\mathbf{0}))}).$$

Proof. Let η be a smooth cut-off function for $B_1(\mathbf{0}) \subset B_2(\mathbf{0})$ and define a new cut-off function by

$$\eta_{x_0,\rho} = \eta\left(\frac{x - x_0}{\rho}\right).$$

Then

$$v = u \cdot \eta_{x_0,\rho} \in C_0^{2,\alpha}(B_{2\rho}(x_0))$$

for $x_0 \in B_1(\mathbf{0})$ and $B_{2\rho}(x_0) \subset B_2(\mathbf{0})$. Furthermore, $v = u$ on $B_\rho(x_0)$.

Let L_0 be the elliptic operator obtained via $A_0 = (a^{ij}(x_0))_{i,j=1}^n$. Let

$$L_0 v = Lv - \sum (a^{ij} - a^{ij}(x_0))\partial_{ij} v.$$

Thus

$$\begin{aligned}[L_0 v]_\alpha &\leq [Lv]_\alpha + \|D^2 v\|_{C^0} \cdot [a^{ij} - a^{ij}(x_0)]_\alpha + \|a^{ij} - a^{ij}(x_0)\|_{C^0(B_{2\rho}(x_0))} \cdot [D^2 v]_\alpha. \\ &\leq [Lv]_\alpha + [a^{ij}]_\alpha \cdot \|v\|_{C^2} + 2\rho^\alpha \cdot [a^{ij}]_\alpha \cdot [D^2 v]_{C^\alpha}.\end{aligned}$$

Also observe that

$$[D^2 v]_\alpha \leq C_1(n, \alpha, \lambda, \Lambda)[L_0 v]_\alpha.$$

Thus choosing ρ small (depending only on gives $(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)$) gives

$$[D^2 v]_\alpha \leq C_1(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)([Lv]_\alpha + \|v\|_{C^2}).$$

Now observe

$$\|v\|_{C^2} \leq C(n) \cdot \|u\|_{C^2} \cdot \|\eta_{x_0,\rho}\| \leq C_2(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)\|u\|_{C^2(B_2(\mathbf{0})}$$

and

$$Lv = Lu \cdot \eta_{x_0,\rho} + u \cdot L\eta_{x_0,\rho} + 2\sum a^{ij}\partial_i u \partial_j \eta_{x_0,\rho}$$

which implies

$$\begin{aligned}[Lv]_\alpha &\leq C_3(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)(\|Lu\|_{C^{0,\alpha}(B_2(\mathbf{0}))} + \|u\|_{C^{1,\alpha}(B_2(\mathbf{0}))}) \\ &\leq C_3(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)(\|Lu\|_{C^{0,\alpha}(B_2(\mathbf{0}))} + \|u\|_{C^2(B_2(\mathbf{0}))})\end{aligned}$$

Finally

$$\begin{aligned}[D^2 u]_{\alpha, B_1(\mathbf{0})} &\leq \sup_{x_0 \in B_1(\mathbf{0})} [D^2 v]_{\alpha, B_1(\mathbf{0})} \\ &\leq C_4(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)(\|Lu\|_{C^{0,\alpha}(B_2(\mathbf{0}))} + \|u\|_{C^2(B_2(\mathbf{0}))}).\end{aligned}$$

To conclude note

$$\begin{aligned}\|u\|_{C^{2,\alpha}(B_1(\mathbf{0}))} &= [D^2 u]_{\alpha, B_1(\mathbf{0})} + \|u\|_{C^2(B_2(\mathbf{0}))} \\ &\leq (1 + C_4(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha))(\|Lu\|_{C^{0,\alpha}(B_2(\mathbf{0}))} + \|u\|_{C^2(B_2(\mathbf{0}))}).\end{aligned}$$

□

Remark. Note that the cut-off function only depends on the distance $d(B_1(\mathbf{0}), \partial B_2(\mathbf{0}))$ so that we can prove a similar estimate for

$$\|u\|_{C^2(\Omega')} \leq C(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha, d(\Omega', \partial\Omega)) \cdot \left(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^2(\Omega)}\right).$$

Theorem 7.8. *For any uniformly elliptic operator with $b^k = c = 0$ there is a constant $C = C(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha) > 0$ such that for all $u \in C^{2,\alpha}(\Omega)$ it holds*

$$\sup_{x\in\Omega} \min\{d(x, \partial\Omega), 1\}^2 |D^2 u(x)| \leq C \left(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}\right)$$

and

$$\sup_{x\in\Omega} \min\{d(x, \partial\Omega), 1\}^{\frac{n}{2}+2} |D^2 u(x)| \leq C \left(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

In particular, for all $\Omega' \subset\subset \Omega$ there is a constant $\tilde{C} = C(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha, d(\Omega', \partial\Omega))$ such that

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq \tilde{C} \cdot \begin{cases} \left(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}\right) \\ \left(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}\right). \end{cases}$$

Proof. The L^2-estimate only depends on an application of Ehrlings Lemma for the inclusion $C^{2,\alpha} \to C^2 \to L^2$ and the scaling property of the L^2-norm, more precisely if $\tilde{u}(x) = u(x_0 + \rho x)$ with $B_{2\rho}(x_0) \subset \Omega$ we get $\|\tilde{u}\|_{L^1(B_1(\mathbf{0}))} = \rho^{\frac{n}{2}} \|u\|_{L^1(B_\rho(x_0))}$. The factor $\rho^{\frac{n}{2}}$ is carried along and induces the addition $d(x, \partial\Omega)^{\frac{n}{2}}$-factor in estimate containing the L^2-norm of u. We leave the details to the reader and only focus on the C^0-estimate.

For this define

$$S = \sup_{x\in\Omega} \min\{d(x, \partial\Omega), 1\}^2 |D^2 u(x)|.$$

Pick $x_0 \in \Omega$ and set $\rho = \min\{\frac{1}{3} d(x_0, \partial\Omega), \frac{1}{3}\} < 1$. Then $B_{2\rho}(x_0) \subset \Omega$ and for all $x \in B_{2\rho}(x_0)$ it holds $\min\{d(x, \partial\Omega), 1\} \geq \rho$. Define a function $\tilde{u} \in C^{2,\alpha}(B_2(\mathbf{0}))$ by $\tilde{u}(x) = u(x_0 + \rho x)$ and observe

$$\|\tilde{u}\|_{C^0(B_2(\mathbf{0}))} \leq \|u\|_{C^0(\Omega)}$$
$$\|D^2 \tilde{u}\|_{C^0(B_1(\mathbf{0}))} = \rho^2 \|D^2 u\|_{C^0(B_\rho(x_0))}$$

and

$$\begin{aligned} \|\tilde{L}\tilde{u}\|_{C^\alpha(B_2(\mathbf{0}))} &= \rho^2 \|Lu\|_{C^0(B_{2\rho}(x_0))} + \rho^{2+\alpha} [Lu]_{\alpha, B_{2\rho}(x_0)} \\ &\leq \|Lu\|_{C^\alpha(\Omega)} \end{aligned}$$

where $\tilde{L}$ is the rescaled elliptic operator which is still uniformly elliptic with the same constants as L and $[\tilde{a}^{ij}]_\alpha \leq [a^{ij}]_\alpha$ thus the previous $C^{2,\alpha}$- estimate holds for $C = C(n, \alpha, \lambda, \Lambda, [a^{ij}]_\alpha)$.

Then

$$\begin{aligned} \min\{d(x_0, \partial\Omega), 1\}^2 |D^2 u(x_0)| &= 9\rho^2 |D^2 u(x_0)| \\ &\leq 9 \|D^2 \tilde{u}\|_{C^0(B_1(\mathbf{0}))} \\ &\leq 9C(\|\tilde{L}\tilde{u}\|_{C^{0,\alpha}(B_2(\mathbf{0}))} + \|\tilde{u}\|_{C^2(B_2(\mathbf{0}))}) \\ &\leq 9C(\|Lu\|_{C^{0,\alpha}(\Omega)} + \epsilon\rho^2 \|D^2 u\|_{C^0(B_{2\rho}(x_0))} + C_{n,\alpha,\epsilon} \|u\|_{C^0(\Omega)}) \end{aligned}$$

where we use Ehrlings Lemma to estimate

$$\|\tilde{u}\|_{C^2(B_2(\mathbf{0})} \leq \epsilon \|D^2 \tilde{u}\|_{C^0(B_2(\mathbf{0}))} + C_{n,\alpha,\epsilon} \|\tilde{u}\|_{C^0(B_2(\mathbf{0}))}.$$

Note that

$$\begin{aligned} \rho^2 \|D^2 u\|_{C^0(B_{2\rho}(\mathbf{0}))} &\leq \sup_{x\in B_{2\rho}(x_0)} \min\{d(x, \partial\Omega), 1\}^2 |D^2 u(x)| \\ &\leq S. \end{aligned}$$

Thus choosing $\epsilon = \frac{1}{18C}$ then

$$\min\{d(x_0,\partial\Omega),1\}^2|D^2u(x_0)| \le \frac{1}{2}S + C_2(n,\alpha,\lambda,\Lambda,[a^{ij}]_\alpha)(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)})$$

which implies the claim

$$S \le 2C_2(n,\alpha,\lambda,\Lambda,[a^{ij}]_\alpha)(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}).$$

□

Remark (Boundary/Global Regularity). Global regularity is obtained by doing the same analysis on $\mathbb{R}^{n,+} == \{x \in \mathbb{R}^n \,|\, x_n \ge 0\}$ and $B_2^+(\mathbf{0}) = B_2(\mathbf{0}) \cap \mathbb{R}^{n,+}$. If the boundary of Ω is sufficiently smooth then we may cover Ω by charts and obtain estimates for a finite cover of Ω which yields the global regularity

$$\|u\|_{C^{2,\alpha}(\Omega)} \le C(n,\alpha,\lambda,\Lambda,[a^{ij}]_\alpha,\Omega)(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)})$$

for $u \in C_0^{2,\alpha}(\Omega)$.

If $u = \varphi$ on $\partial\Omega$ for some $\varphi \in C^{2,\alpha}(\Omega)$ then we also get

$$\|u\|_{C^{2,\alpha}(\Omega)} \le C(n,\alpha,\lambda,\Lambda,[a^{ij}]_\alpha,\Omega)(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\Omega)}).$$

8. Method of Continuity

Lemma 8.1 (A priori bounds)**.** *Let L be an elliptic operator with $c \le 0$ on a bounded domain Ω such that $\beta := \sup \frac{|b^k|}{\lambda} < \infty$. Then for all $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Lu \ge f$ it holds*

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+ + \frac{C}{\lambda}\sup_\Omega |f^-|$$

where $C = e^{(\beta+1)\operatorname{diam}\Omega} - 1$.

Proof. Set $L_0 = \sum a^{ij}\partial_{ij} + \sum b^k\partial_k$ then for $\alpha \ge \beta + 1$ and $d = \inf_{x\in\Omega} x_1$ it holds

$$L_0 e^{\alpha(x_1-d)} = (\alpha^2 a^{11} + \alpha b^1)e^{\alpha(x_1-d)} \ge \lambda\alpha(\alpha-\beta)e^{\alpha(x_1-d)} \ge \lambda \quad \text{in } \Omega.$$

Set

$$v = \sup_{\partial\Omega} u^+ + (e^{\alpha\operatorname{diam}\Omega} - e^{\alpha(x_1-d)})\sup_\Omega \frac{|f^-|}{\lambda} \ge 0$$

Then

$$Lv = L_0 v + cv \ge L_0 v \le -\lambda \sup_\Omega \frac{|f^-|}{\lambda}$$

so that

$$L(v-u) \le -\lambda\left(\sup_\Omega \frac{|f^-|}{\lambda} + \frac{f}{\lambda}\right) \le 0 \quad \text{in } \Omega.$$

Since $(v-u)\big|_{\partial\Omega} \ge \sup_{\partial\Omega} u^+ - u|_{\partial\Omega} \ge 0$, the minimum principle applied to $v-u$ implies $v - u \ge 0$ in Ω. Thus we have

$$\sup_\Omega u \le \sup_\Omega v \le \sup_{\partial\Omega} u^+ + \left(e^{(\beta+1)\operatorname{diam}\Omega} - 1\right)\sup_\Omega \frac{|f^-|}{\lambda}.$$

□

Corollary 8.2. *If $Lu = f$ then $\sup_\Omega |u| \le \sup_{\partial\Omega}|u| + \frac{C}{\lambda}\sup_\Omega |f|$.*

Lemma 8.3 (Banach Fixed Point Theorem)**.** *Let $\Psi : X \to X$ be a map on a Banach space $(X, \|\cdot\|)$ to itself. If*

$$\|\Psi(x) - \Psi(y)\| \le K\|x - y\|$$

for some $K < 1$ then there is a unique $x_0 \in X$ such that $\Psi(x_0) = x_0$.

Proof. The condition implies that Ψ is K-Lipschitz. Since $K < 1$ there can be at most one fixed point. To obtain existence, observe that for a fixed $y_0 \in X$ the sequence $(\Psi^n(y_0))_{n\in\mathbb{N}}$is Cauchy and by continuity of Ψ converges a fixed points. □

Proposition 8.4 (Continuity Method)**.** *Assume $L_0, L_1 : X \to Y$ are two bounded linear maps between two Banach spaces such that for some $c > 0$ it holds*

$$\|L_t x\|_Y \ge c\|x\|_X$$

where $L_t = (1 - t)L_0 + tL_1$ and $t \in [0, 1]$. Then L_0 is surjective if and only if L_1 is surjective.

Proof. Assume for some $s \in [0, 1]$ the operator L_s is surjective. The condition implies that L_s must be also injective.

Let $t \in [0, 1]$. Then the equation $L_t x = y$ is equivalent to

$$L_s x + (s - t)(L_1 - L_0)x = y.$$

Since L_s is bijective this is equivalent to

$$x = (s - t)L_s^{-1}(L_1 - L_0)x + L_s^{-1}y = \Psi_{s,t,y}(x).$$

Now

$$\begin{aligned}\|\Psi_{s,t,y}(x) - \Psi_{s,t,y}(y')\| &\le |s - t|\|L_s^{-1}\| \cdot \|L_1 - L_0\| \\ &\le c^{-1}|s - t|(\|L_0\| + \|L_0\|)\end{aligned}$$

which implies that $\Psi_{s,t,y}$ has a unique fixed point if

$$|s - t| < \frac{c}{\|L_0\| + \|L_1\|} = \tilde{c}.$$

In particular, if $t \in (s - \tilde{c}, s + \tilde{c}) \cap [0, 1]$ then L_t is surjective as this bound is independent of $y \in Y$ and the fixed point satifies $L_t x = y$. Inductively we can show that L_t is surjective if $t \in (s - (n + \frac{1}{2})\tilde{c}, s + (n + \frac{1}{2})\tilde{c}) \cap [0, 1]$ and L_s is surjective. Since $[0, 1]$ is bounded we immediately get the result. □

If L is uniformly elliptic then for $u \in W_0^{1,2}(\Omega)$ we have

$$\lambda \int u^2 dx \le \lambda\|u\|^2_{W^{1,2}} \le B_L(u, u) = \int f u dx \le \|f\| \cdot \|u\|_2$$

so that the global $W^{2,2}$-estimate implies

$$\|u\|_{W^{2,2}} \le C\|f\|_2$$

where C depends on the ellipticity constants and the Lipschitz norm of the coefficients. Then

$$L_t = (1 - t)\Delta + tL$$

would satisfy the $W^{2,2}$-bounds with the same constant. In particular, we would know that L is surjective between $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $L^2(\Omega)$ if and only if the Laplace operator Δ is surjective between those spaces.

Note that such a result already follows from the general existence theory of uniformly elliptic operators. In the following we want to obtain a similar existence result for uniformly elliptic operators satisfying the Schauder estimates. Assume Ω has C^∞-boundary. Using the a priori estimate and the Schauder estimate we also obtain for $u \in C_0^{2,\alpha}(\Omega)$

$$\|u\|_{C^{2,\alpha}(\Omega)} \le C\|Lu\|_{C^\alpha(\Omega)}$$

which shows that $L : C_0^{2,\alpha}(\Omega) \to C_0^\alpha(\Omega)$ satisfies the assumption of the Method of Continuity. Note that the coefficients of L need only to be Hölder continuous so that we cannot use the $W^{2,2}$-estimate which need Lipschitz coefficients.

To see that Δ is surjective let $f \in C^\alpha(\Omega)$ be given and $f_n \to f$ in $C_0^\alpha(\Omega)$ with $f_n \in C_c^\infty(\Omega)$. The Sobolev theory gives a sequence $u_n \in W_0^{1,2}(\Omega)$ with $\Delta u_n = f_n$ and the regularity theory shows $u_n \in C^\infty(\Omega)$. Now the bound above shows

$$\|u_n\|_{C^{2,\alpha}(\Omega)} \le C\|f_n\|_{C^\alpha(\Omega)}$$

which shows by Arzela–Ascoli that $u_{n'} \to u$ in C^2-norm with $u \in C^{2,\alpha}(\Omega)$. In particular, $\Delta u = f$. By the Method of Continuity there is also a $\tilde{u} \in C_0^{2,\alpha}(\Omega)$ with $L\tilde{u} = f$.

Appendix A. Topology

Recall that a topological space is a tuple (X, τ) (e.g. $\mathbb{R}^n$ with its open sets) such that $\tau \subset 2^X$, the set of *open subset*s, satisfies $\varnothing, X \in \tau$, $U \cap V \in \tau$ for all $U, V \in \tau$ and whenever $U_i \in \tau$ for an index set $i \in I$ then also $\cup_{i\in I} U_i \in \tau$. Any set $C \subset X$ such that $X\backslash C$ is open, will be called *closed*. Note that $\cap_{i\in I} C$ is closed whenever each C_i, $i \in I$, is closed.

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to *converge* to x, denoted by $x = \lim_{n\to\infty} x_n$, if all *open neighborhood U of x*, i.e. $U \in \tau$ with $x \in U$, there is an $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \ge n_0$.

In general, convergence of sequences is NOT enough to describe the topology completely. For this one needs the concepts of *nets*: A net $(x_i)_{i\in I}$ in X is “subset” $\{x_i\}_{i\in I}$ of X that is indexed by a directed set $(I, \ge)$. A *directed set* $(I, \ge)$ is a *partially order set* ($\ge$ is reflexive and transitive) such that any two elements have an upper bound, i.e. for each $a, b \in I$ there is a $c \in I$ such that $c \ge a$ and $c \ge b$.

Now we say that the net $(x_i)_{i\in I}$ converges to x, written $x = \lim_{i\in I} x_i$ if for open neighborhoods U of x there is an $i_0 \in I$ such that $x_i \in U$ for all $i \ge i_0$.

If $A \subset X$ is any subset then we define the *interior* $\operatorname{int} A$ of A and the *closure* $\operatorname{cl} A$ of A as follows

$$\operatorname{int} A = \bigcup_{U\subset A, U\in\tau} U = \{x \in A \mid \exists U \text{ open} : x \in U \subset A\}$$

$$\operatorname{cl} A = \bigcap_{A\subset C, X\backslash C\in\tau} C = \{x \in X \mid \exists \text{net } (x_i)_{i\in I} : x = \lim_{i\in I} x_i\}.$$

In case of brevity we sometimes[11] write $\operatorname{cl} A = \bar{A}$ and $\operatorname{int} A = \mathring{A}$.

A subset $A \subset B$ be *dense* in B if for each $x \in B$ there is a net $(x_i)_{i\in I}$ in A converging to B. This is equivalent to saying that the closure of A and B in X is the same. The topological space (X, τ) is called *separable* if it admits a countable subset D which is dense in X.

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[11] Be aware that for metric spaces the *closed ball*$\bar{B}_r(x)$ is not necessarily equivalent to the closure $\operatorname{cl}(B_r(x))$, hence the notation $\bar{A}$ will be avoided if possible!

Also a set $A \subset X$ is *compact*[12] if whenever $A \subset \cup_{i\in I} U_i$ for open sets $\{U_i\}_{i\in I}$ then there is a finite subset $I' \subset I$ such that $A \subset \cup_{i\in I'} U_i$. One can show that any compact set has to be closed. Using the concept of nets one can show that a set A is compact if every net in A admits a convergent subnet with limit in A. Note however that a set B where every sequences in B admits a convergent subsequence with limit in A is, in general, not compact. Sets satisfying this condition will be called *sequentially compact*.

Let Ω be an open set of a topological space X. Then for a subset $A \subset X$ we say A is compactly contained in Ω, written as $A \subset\subset \Omega$, if

$$\mathrm{cl}(A) \subset \Omega$$

and $\mathrm{cl}\, A$ is compact.

A map $f : X \to Y$ between topological spaces (X, τ) and (Y, τ') is said to be *continuous* if for all $V \in \tau'$ it holds $f^{-1}(V) \in \tau$. The function f has *compact support* in Ω if $\mathrm{supp}\, f \subset\subset \Omega$.

A *metric*[13] d on X is a symmetric, positive definite function on $X \times X$ that satisfies the triangle inequality, i.e. for all $x, y, z \in X$ it holds $d(x, z) \le d(x, y) + d(y, z)$.

Given a metric d on a set X there is a natural topology τ_d on X induced by d:

$$\tau_d = \{U \in 2^x \mid \forall x \in U \exists r > 0 : B_r(x) \subset U\}.$$

We call the tuple (X, d) a metric space.

Note that if $A \subset\subset \Omega$ then the function

$$x \mapsto d(x, \partial\Omega) = \inf\{d(x, y) \mid y \in \partial\Omega\}$$

is uniformly bounded away from zero on the set A.

The convergence with respect to metric topology τ_d is equivalent to the following: $x_i \to x$ in τ_d for a net $(x_i)_{i\in I}$ iff

$$\forall \epsilon > 0 \exists i_0 \in I \forall i \ge i_0 : d(x, x_i) < \epsilon.$$

This also show that the topology τ_d can descripted entirely by sequences instead of nets.

The metric also allows one to define the concept of *Cauchy sequence*, i.e. $(x_n)_{n\in\mathbb{N}}$ is Cauchy if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n, m \ge N : d(x_n, x_m) < \epsilon.$$

Using the triangle inequality we can show that a converging sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. The converse is in general not true. A metric space (X, d) for which every Cauchy sequence is convergent will be called *complete*. Hence in a complete metric space the concept of Cauchy sequences and convergent sequences is equivalent.

If a metric space (X, d) is not complete then there is a (unique up to isomorphism) completion $(\tilde{X}, \tilde{d})$ such that (X, d) embeds isometrically in $(\tilde{X}, \tilde{d})$ such that the image of X in $\tilde{X}$ is dense. For that reason we often regard X as a subset of the completion $\tilde{X}$.

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[12] *In a complete metric space (e.g. $(\mathbb{R}^n, \|\cdot - \cdot\|)$) this is equivalent to "every sequence in A has a subsequence converging to a point in A".*

[13] e.g. on $\mathbb{R}^n$ take any norm $\|\cdot\|$ and define $d(x, y) = \|x - y\|$.

Appendix B. Measure and Integration Theory

A *measurable space* $(X,\mathcal{B})$ is a set X equipped with a *σ-algebra* $\mathcal{B}\subset 2^X$, i.e. $\mathcal{B}$ contains the empty set and is closed under countable unions and complements. If (X,τ) is a topological space then the *Borel σ-algebra* $\mathcal{B}(X)$ is the smallest σ-algebra containing all open set $U\in\tau$. A set in $A\in\mathcal{B}(X)$ will be call *(Borel) measurable*.

A map $\mu:\mathcal{B}(X)\to[0,\infty]$ is called a *Borel measure* if it is monotone, σ-additive and $\mu(\varnothing)=0$. The statement "a property $\mathfrak{P}(x)$ holds for μ-almost all $x\in X$" is a short description for the following: there is a measurable set Ω such that $\mu(X\backslash\Omega)=0$ and property $\mathfrak{P}(x)$ holds for all $x\in\Omega$.

A function $f:X\to\mathbb{R}$ is called *measurable* if for all Borel set $A\in\mathcal{B}(\mathbb{R})$ the set $f^{-1}(A)$ is measurable. A measurable function is simple if f is of the form

$$f=\sum_{i\in\mathbb{N}}a_i\chi_{A_i}$$

for $a_i\in\mathbb{R}^n$ and disjoint Borel set $\{A_i\}_{i\in\mathbb{N}}$.

For simple functions $f:X\to[0,\infty)$ define

$$\int fd\mu:=\sum_{i\in\mathbb{N}}a_i\mu(A_i).$$

If $f:X\to[0,\infty)$ is measurable then define

$$\int fd\mu=\sup\{\int\tilde{f}d\mu\,|\,\tilde{f}:X\to[0,\infty)\text{ is a simple measurable function with }\tilde{f}\le f\}.$$

It is know that there is a sequence $f_n:X\to[0,\infty)$ of simple functions such that $f_n\le f_{n+1}$ and for μ-almost all $x\in X$ it holds $f(x)=\lim_{n\to\infty}f_n(x)$, i.e. f_n converges μ-almost every to f. In that case it can be shown

$$\int fd\mu=\lim_{n\to\infty}\int f_nd\mu.$$

More generally, the *Theorem of Monotone Convergence* says that whenever $f_n: X\to[0,\infty)$ is a non-decreasing sequence measurable functions converging almost everywhere to f then it holds $\int fd\mu=\lim_{n\to\infty}\int f_nd\mu$. Note that f is measurable as it is the μ-almost everywhere limit of measurable functions.

Fatou's Lemma says that if for a sequence $f_n:X\to[0,\infty)$ of measurable function it holds $f(x)=\liminf_{n\to\infty}f_n(x)$ for μ-almost all $x\in X$ then f is measurable and

$$\int fd\mu\le\liminf_{n\to\infty}\int f_nd\mu.$$

We say a measurable function $f:X\to\mathbb{R}$ is *μ-integrable* if $\int f_\pm d\mu<\infty$, or equivalently $\int|f|d\mu<\infty$. Denote the space of μ-integrable functions by $L^1(\mu)$. The assignment $f\mapsto\int|f|d\mu$ induces a norm on $L^1(\mu)$ and makes it into a *complete Banach space*. Furthermore, we define

$$\int fd\mu=\int f_+d\mu-\int f_-d\mu.$$

The *Theorem of Dominated Convergence* says if a sequence of μ-integrable functions $f_n:X\to\mathbb{R}$ converges μ-almost everywhere to $f:X\to\mathbb{R}$ and $|f_n|\le g$ for a μ-integrable function then f is μ-integrable and $\int fd\mu=\lim_{n\to\infty}\int f_nd\mu$.

On $\mathbb{R}^n$ equipped with a norm $\|\cdot\|$ there is a unique Borel measure λ^n, called the Lebesgue measure (associated to $(\mathbb{R}^n, \|\cdot\|)$) that is translation invariant[14] and satisfies

$$\lambda^n(B_1(0)) = \omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Note that λ^n depends on the norm $\|\cdot\|$. However, the Lebesgue measures associated to two given norm different by a (multiplicative constant). If one chooses the norm induced by the standard scalar product on $\mathbb{R}^n$ the measure λ^n is the "usual" Lebesgue measure.

For each open subset $\Omega \subset \mathbb{R}^n$ the Lebesgue measure induces a natural measure $\lambda^n\big|_\Omega$ (usually also denoted by λ^n) by restricting λ^n to subsets of Ω. We denote the space of λ^n-integrable functions on Ω by $L^1(\Omega)$.

If $\Omega \subset \mathbb{R}^n$ is open and bounded and $\partial\Omega$ is "sufficiently" then there is a natural measure λ^{n-1} on $\partial\Omega$ such that for all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ it holds

$$\int f dz = \int f d\lambda^{n-1} = \lim_{\epsilon\to 0} \int_{(\partial\Omega)_\epsilon} f d\lambda^n.$$

Again the measure λ^{n-1} depends on the norm (resp. metric) chosen on $\mathbb{R}^n$.

Appendix C. Polar coordinates.

Let $f : \mathbb{R}^n \to \mathbb{R}$ an integrable function with compact support (e.g. f is a bounded function) then

$$\int f(x)dx = \int_0^r \int_{\partial B_r(0)} f(z)dzdr.$$

If we observe that for each $z \in \partial B_r(0)$ there is a unique $\omega \in \partial B_1(0) = \mathbb{S}^{n-1}$ such that $z = r\omega$ and that

$$|\partial B_r(0)| = r^{n-1}|\partial B_1(0)|$$

we immediately the formula for polar coordinates, i.e.

$$\int f(x)dx = \int_0^r \int_{\mathbb{S}^{n-1}} f(r\omega)r^{n-1}d\omega dr.$$

Furthermore, we can translate $x \mapsto x - a$ to also obtain the following

$$\int f(x)dx = \int_0^r \int_{\mathbb{S}^{n-1}} f(a + r\omega)r^{n-1}d\omega dr.$$

Appendix D. List of definition/symbols

For a topological space (X, τ) and $A \subset X$:

$$\operatorname{int} A = \mathring{A} = \bigcup_{A\supset U \text{ open}} U$$
$$\operatorname{cl} A = \bar{A} = \bigcap_{A\subset C \text{ closed}} C.$$

[14] $\lambda^n(A) = \lambda^n(A + x)$ for all measurable set A and $x \in \mathbb{R}^n$

If (X, d) is a metric space and $A \subset X$ then

$$B_r(x) = \{y \in X \mid d(x,y) < r\}$$
$$\bar{B}_r(x) = \{y \in X \mid d(x,y) < r\}$$
$$A_\epsilon = \text{"}\epsilon - \text{neighrborhood of } A\text{"} = \bigcup_{x \in A} B_\epsilon(x).$$

The function $\chi_A : X \to \mathbb{R}$ is defined as

$$\chi_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A. \end{cases}$$

For functions $f : X \to \mathbb{R}$ and measures μ (on a given σ-algebra):

$$\fint_A f d\mu = \frac{1}{\mu(A)} \int_A f d\mu = \frac{\int_A f d\mu}{\int_A d\mu}$$
$$\{f \circledast c\} = \{x \in X \mid f(x) \circledast c\} \quad \text{for } \circledast \in \{<, >, \geq, \leq, =, \neq\}$$
$$\operatorname{supp} f = \operatorname{cl}\{f(x) \neq 0\}$$
$$f_+ = \chi_{\{f>0\}} \cdot f$$
$$f_- = -\chi_{\{f<0\}} \cdot f$$
$$f = f_+ - f_-$$
$$|f| = f_+ + f_-$$
$$L^p(\mu) = \{f : X \to \mathbb{R} \mid \int |f|^p d\mu < \infty\}$$
$$\operatorname{ess\,sup}_\mu f = \sup\{r \in \mathbb{R} \mid \mu(\{f > r\}) > 0\}$$
$$\operatorname{ess\,inf}_\mu f = -\operatorname{ess\,sup} f.$$

For functions $f : \mathbb{R}^n \to \mathbb{R}$ and the n-dimensional Lebesgue measure λ^n and measurable sets $A \subset \mathbb{R}$

$$\int_A f dx = \int_A f d\lambda^n$$
$$\operatorname{ess\,sup} f = \operatorname{ess\,sup}_{\lambda^n} f$$
$$\operatorname{ess\,sup}_\Omega f = \operatorname{ess\,sup}_{\lambda^n|_\Omega} f.$$
$$L^p(\Omega) = L^p(\lambda^n|_\Omega) = \begin{cases} \{f : \Omega \to \mathbb{R} \mid \int_\Omega |f|^p dx < \infty\} & p \in (0, \infty) \\ \operatorname{ess\,sup}_\Omega |f| < \infty & p = \infty \end{cases}$$

If $\Omega \subset \mathbb{R}^n$ is open and bounded with $\partial\Omega$ “nice” then

$$\int_{\partial\Omega} f dz = \int_{\partial\Omega} f d\lambda^{n-1} \left(= \text{"} \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{(\partial\Omega)_\epsilon} f dx \text{"} \right).$$