EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Faculty of Science

Department of Mathematics

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Linear PDE

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Project 1

Exercise (Perron's Method for general solution operators)
(15 points *) This project is optional.

Assume $\Omega \subset \mathbb{R}^n$ is an connected open set. Assume the following concepts are given.

Solution operator

Given any ball $B_r(x) \subset\subset \Omega$ there is a *linear* operator $P_B : C^0(\partial B_r(x)) \to C^0(\bar{B}_r(x))$ such that given $g \in C^0(\partial B_r(x))$ and $h := P_{B_r(x)}g$ it holds

- $h|_{\partial B_r(x)} = g$.
- if $g \equiv c$ for some $c \in \mathbb{R}$ then
$$h \equiv c \quad \text{on } \bar{B}_r(x).$$
- h satisfies the *Harnack inequality* on $\Omega^+ = \operatorname{int}\{h \geq 0\}$, i.e. for $\Omega' \subset\subset \Omega^+$ there is a constant $C > 0$, not depending on h, such that
$$\sup_{\Omega'} h \leq C \inf_{\Omega'} h.$$

Weak subharmonicity

A function $u \in C^0(\Omega)$ is called *weakly subharmonic* if for all $B_r(x) \subset\subset \Omega$ it holds
$$u \leq P_B g \quad \text{on } \bar{B}_r(x)$$
whenever $u|_{\partial B_r(x)} \leq g$. If both u and $-u$ are weakly subharmonic then u is called *weakly harmonic*.

Prove the following theorem.

Theorem (Perron's Method). *Let $g \in C^0(\partial\Omega)$ and define*
$$S_- = \{u \in C^0(\bar{\Omega}) \,|\, u \text{ is weakly subharmonic and } u|_{\partial\Omega} \leq g\}.$$
Then the function u_* defined by
$$u_*(x) = \sup_{u \in S_-} u(x)$$
is weakly harmonic in Ω.

In order to use the proof of Perron's theorem for mean-value harmonic functions it might help to prove the following.

- the set of weakly harmonic functions is a vector spaces

- any harmonic function satisfies the strong minimum principle (and thus the strong maximum principle)
- a weakly subharmonic function satisfies the maximum principle
- Harnack's Convergence Theorem holds for weakly harmonic functions
- if u_1 and u_2 are weakly subharmonic then $\max\{u_1, u_2\}$ is weakly subharmonic
- the Replacement Lemma holds, i.e. if u is subharmonic in Ω and $B \subset\subset \Omega$ is a ball then

$$\tilde{u}(x) = \begin{cases} P_B(u\big|_{\partial B})(x) & x \in \bar{B} \\ u(x) & x \in \Omega \backslash \bar{B} \end{cases}$$

  is also subharmonic.

Each of those can be obtained by slight variations of the proofs of corresponding statements in the lecture notes and by the solution of **Exercise** 12.

Deadline 12.06.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.