EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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Department of Mathematics

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Linear PDE

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Exercise sheet 1

Exercise 1

(1 point) Let $A : \Omega \to \mathbb{R}^{n\times n}$ be a matrix-valued map and $A_{\text{sym}} = \frac{1}{2}(A + A^T)$ be its symetrization. Show that the operators

$$Lu(x) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_i\partial_j u(x)$$

and

$$\tilde{L}u(x) = \sum_{i,j=1}^{n} a^{ij}_{\text{sym}}(x)\partial_i\partial_j u(x)$$

are equivalent for all $u \in C^2(\Omega)$.

Exercise 2

(2 points) Let $u \in C^0(\Omega)$ be a continuous function. If Ω is open and connected, and for all balls $B_r(x) \subset \Omega$ it holds $u(y) = u(x)$ whenever $y \in B_{\frac{r}{2}}(x)$ then u is constant in Ω, in other words, prove that a locally constant function on a connected domain is constant.

Exercise 3

(2 points) Let $C^0_c(\Omega)$ be the space of continuous functions $\varphi : \Omega \to \mathbb{R}$ with support

$$\operatorname{supp}\varphi = \operatorname{cl}\{x \in \Omega \,|\, \varphi(x) > 0\}$$

compactly contained in Ω (short $\operatorname{supp}\varphi \subset\subset \Omega$). Show that a function $w \in C^0(\Omega)$ is non-negative if and only if for all non-negative function $\varphi \in C^0_c(\Omega)$ it holds

$$\int w\varphi dx \geq 0.$$

(2 points *)[1] Can you show the same result assuming $w \in L^1(\Omega)$?

Exercise 4

(2 points) Let $u \in C^2(\bar{\Omega})$ and assume for all non-negative $\varphi \in C^\infty_c(\Omega)$, i.e. all smooth functions φ with support $\operatorname{supp}\varphi \subset\subset \Omega$, it holds

$$\int_\Omega |\nabla u|^2 dx \leq \int_\Omega |\nabla(u + \varphi)|^2 dx.$$

Then u is superharmonic in Ω, i.e. it holds $-\Delta u \geq 0$ in Ω. What can be said about Δu if the inequality above holds for all $\varphi \in C^\infty_c(\Omega)$?

[1] Exercises with a * are optional.

Exercise 5

(Running exercise) In the exercise sheet a weak notion of convergence for certain classes of Banach spaces, i.e. complete normed spaces of possible infinite dimension, will be developed and depending essentially only on completeness and uniform convexity of the norm it will be shown that bounded sequences have weakly convergent subsequences.

(a) (3 points) That for $x, y \in \mathbb{R}$ and $p \geq 2$ it holds

$$|x_+|^p + |x_-|^p \leq \frac{1}{2}(|x|^p + |y|^p)$$

where $x_\pm = \frac{x \pm y}{2}$.
(Hint: Show and use for $a, b \geq 0$ it holds $a^{\frac{p}{2}} + b^{\frac{p}{2}} \leq (a+b)^{\frac{p}{2}}$, for $p = 2$ this is an equality, and $r \mapsto |r|^{\frac{p}{2}}$ is convex).
Infer that this holds also in $L^p(\Omega)$ with $p \geq 2$, i.e. if $f, g \in L^p(\Omega)$ then

$$\|\frac{f+g}{2}\|^p_{L^p(\Omega)} + \|\frac{f-g}{2}\|^p_{L^p(\Omega)} \leq \frac{1}{2}\left(\|f\|^p_{L^p(\Omega)} + \|g\|^p_{L^p(\Omega)}\right),$$

i.e. the L^p-norm is p-uniformly convex.

(b) (3 points *) Show that for $p \in (1, 2)$ the L^p-norm cannot be p-uniformly convex as the opposite inequality holds, i.e. for all $f, g \in L^p(\Omega)$ it holds

$$\|\frac{f+g}{2}\|^p_{L^p(\Omega)} + \|\frac{f-g}{2}\|^p_{L^p(\Omega)} \geq \frac{1}{2}\left(\|f\|^p_{L^p(\Omega)} + \|g\|^p_{L^p(\Omega)}\right)$$

which means that the L^p-norm is called p-uniformly smooth.

Deadline 02.05.2016 in mailbox C-buidling 3rd floor or office C5P05/08.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.