EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Faculty of Science

Department of Mathematics

Dr. Martin Kell
Jason Ledwidge

# Linear PDE

**Summer semester 2017** **03.07.2016**

## Exercise sheet 10

**Exercise 34**

(4 points) Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ with $n > 2$. Assume for an $L^2$-function $u$ all its weak derivatives $g_{ij}$, $i, j \in \{1, \ldots, n\}$, are zero. Show that $u$ is a smooth affine function, i.e. $u((1-t)x + ty) = (1-t)u(x) + tu(y)$.

**Exercise 35**

Let $L$ be a uniformly elliptic operator in divergence form with $a^{ij}, b^k, c \in C^\infty(\bar{\Omega})$ on a convex domain $\Omega$.

(a) (3 points) Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with $g(0) = 0$ and bounded derivatives of all order. Show that any solution $u \in W_0^{1,2}(\Omega)$ to the equation

$$Lu = g(u)$$

is smooth.

(b) (3 points) Assume $b^k, c \equiv 0$ and suppose $-Lu = \lambda_{\text{min}} \cdot u$ for $u \in W_0^{1,2}(\Omega)$ and

$$\lambda_{\text{min}} = \inf_{\tilde{u} \in W_0^{1,2}(\Omega)} \frac{B_L(\tilde{u}, \tilde{u})}{\langle \tilde{u}, \tilde{u} \rangle_{L^2}} > 0.$$

Show that $v = |u|$ is also smooth and satisfies $-Lv = \lambda_{\text{min}} \cdot v$. Conclude that $u$ is either positive or negative and the set of all $v \in W_0^{1,2}(\Omega)$ satisfying $-Lv = \lambda_{\text{min}} \cdot v$ is 1-dimensional, i.e. first (non-trivial) eigenvalue is simple.
(Hint: Recall that $|u| \in W^{1,2}(\Omega)$ if $u \in W^{1,2}(\Omega)$.)

**Exercise 36**

(4 points *) Let $L_0$ and $L_1$ be two bounded linear operators between two Banach space $X$ and $Y$. For $t \in [0,1]$ define

$$L_t = (1-t)L_0 + tL_1.$$

Assume, in addition, there is a $c > 0$ not depending on $t$ such that

$$c\|u\|_X \leq \|L_t u\|_Y$$

for all $u \in X$ and $t \in [0,1]$, show that $L_1$ is onto if and only if $L_0$ is onto.

**Deadline 10.07.2016 in before the lecture.**
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.