EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Faculty of Science

Department of Mathematics

Dr. Martin Kell
Jason Ledwidge

Linear PDE

Summer semester 2017 **10.07.2016**

Exercise sheet 11

Exercise 37

Let L be a uniformly elliptic operator with $b^k = c = 0$ on a bounded, connected domain $\Omega \subset \mathbb{R}^n$. Recall that a function $u \in W^{1,2}(\Omega)$ is a weak subsolution to the equation $Lu = f$ (short "$Lu \geq f$ weakly") if for all $v \in W^{1,2}(\Omega)$ with$v \geq 0$ it holds

$$B_L(u,v) \geq -\int_\Omega f \cdot v dx.$$

(a) (3 points) Show that if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,2}(\Omega)$ such that $Lu_n \geq -a_n$ weakly for a sequence $a_n \to 0$ then any $W^{1,2}$-weak limit u of $(u_n)_{n\in\mathbb{N}}$ satisfies $Lu \geq 0$ weakly.

(b) (3 points) Let u be a Lipschitz function on $\bar{\Omega}$ satisfying $Lu \geq f$ weakly in Ω. Then for any $\psi \in C^2(\operatorname{cl} u(\Omega))$ it holds

$$L\psi(u) \geq f \cdot \psi'(u) + \psi''(u) \cdot \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u.$$

In particular, if ψ is convex then $L\psi(u) \geq f \cdot \psi'(u)$.
(Hint: Prove $\psi'(u) \cdot v \in W_0^{1,2}(\Omega)$ for $v \in W_0^{1,2}(\Omega)$ and apply product rule to $\int \sum_{i,j=1}^n a^{ij}\partial_i u \cdot \partial_j(\psi'(u) \cdot v)dx$.)

(c) (2 points *) Show the same result for $\psi \in C^{1,1}(\operatorname{cl} u(\Omega))$ and $u \in W^{1,4}(\Omega)$.

(d) (4 points) Let $C > 0$ be a large constant and $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of Lipschitz functions with Lipschitz constant bounded by C such that $\varphi_n(x) \in (C^{-1}n, Cn)$ and $(\varphi_n(x) - n)_{n\in\mathbb{N}}$ is decreasing for all $x \in \Omega$. In addition, assume $L(\varphi_n^2) \leq C$.
Show that $u = \lim_{n\to\infty} \varphi_n - n$ is a Lipschitz function in $W^{1,2}(\Omega)$ satisfying $Lu \geq 0$ weakly.
(Hint: Apply the exercise above to $\psi(u_n) = \varphi_n$ where $\psi(r) = -\sqrt{r}$ and $u_n = \varphi_n^2$.)

(e) (2 points *) Assume $(\psi_n)_{n\in\mathbb{N}}$ is another sequence satisfying similar assumptions as $(\varphi_n)_{n\in\mathbb{N}}$. Show that if $\varphi_n + \psi_n \leq 2n$ and $\psi_n(x_0) + \varphi_n(x_0) = 2n$ for some $x_0 \in \Omega$ then $Lu = 0$ weakly where u is given as above. In particular, if $a^{ij} \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.

Deadline 17.07.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.