EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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Linear PDE

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Exercise sheet 12

Exercise 40

(3 points) Let L be a uniformly elliptic operator on $\mathbb{R}^n$ with $b^k \equiv c \equiv 0$ and $Lu = 0$ weakly for $u \in W^{1,2}_{loc}(\mathbb{R}^n)$. Show that there is a constant $C = C(n, \lambda, \Lambda) > 2$ such that for all $R > 0$ it holds

$$\int_{B_R(x_0)} |\nabla u|^2 dx \le \frac{C^2}{R^2} \int_{B_{2R}(x_0)} u^2 dx.$$

(Hint: Choose the cut-off function $\eta_{x_0,R}$ with $|\nabla \eta| \le \frac{C}{R}$ and argue as in the proof of the Cacciopoli inequalty).

(2 points *) Use the inequality to show that $u \equiv 0$ if u has finite L^2-norm on $\mathbb{R}^n$.

Exercise 41

Let u be a harmonic function on $\mathbb{R}^n$.

(a) (3 points) Show that for all $x_0 \in \mathbb{R}^n$ and $R > 0$ it holds

$$\sup_{B_R(x_0)} |\nabla u| \le \frac{2^n \cdot C}{R} \sup_{B_{4R}(x_0)} |u|.$$

(Hint: Use Exercise 8 (e), the mean-value property for subharmonic functions and the previous inequality to obtain an estimate of $|\nabla u|^2(y)$ for $y \in B_R(x_0)$.)

(b) (2 points) Use this to show that any harmonic function on $\mathbb{R}^n$ satisfying

$$|u(x)| \le D(1 + \|x\|)$$

for some $D > 0$ is linear, i.e. u is of the form $x \mapsto \langle b, x\rangle + c$.
(Hint: Use Exercise 14 [Liouville's Theorem] to show $(\partial_i u)_{i=1}^n \equiv \mathsf{const}$ and conclude $b \equiv (\partial_i u)_{i=1}^n$)

(c) (1 point *) Conclude that a sublinearly growing harmonic function must be constant.

(d) (3 points *) Show that any harmonic function on $\mathbb{R}^n$ of at most polynomial growth of order k (see below) must be a polynormial of order at most k, i.e. $D^k u \equiv \mathsf{const}$.

Definition. A function has at most *polynomial growth* of order k if

$$D_{u,k} := \lim_{r\to\infty} \sup_{\|x\| \ge r} \frac{|u(x)|}{1 + \|x\|^k} < \infty.$$

If $D_{u,1} = 0$ then it is said to have *sublinear growth.*

Exercise 42

Let L be a uniformly elliptic operator with $b^k \equiv c \equiv 0$ and $\Omega' \subset\subset \Omega$. From the Schauder estimates we know that there is a constant $C = C(n, \lambda, \Lambda, [a^{ij}]_\alpha, d(\Omega', \partial\Omega)) > 0$ such that

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|Lu\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}).$$

(a) (2 points) Show if $(a^{ij})_{i,j=1}^n$ is constant then $Lu \in C^{k,\alpha}(\Omega)$ implies $u \in C^{k+2,\alpha}(\Omega')$.
(Hint: Bound the $C^{2,\alpha}$-norm of $\partial_i^h u$ for $|h| < \frac{1}{2}d(\Omega', \partial\Omega)$ and use Arzela–Ascoli to conclude $\partial_i u \in C^{2,\alpha}(\Omega')$).

(b) (3 points *) Show that there is a constant $C_k = C_k(n, \lambda, \Lambda, \|a^{ij}\|_{k,\alpha}, d(\Omega', \partial\Omega)) > 0$ such that

$$\|u\|_{C^{k+2,\alpha}(\Omega')} \leq C_k \|Lu\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}).$$

Deadline 24.07.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.