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Linear PDE

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Exercise sheet 2

Exercise 6

(1 point) Let $u \in C^2(\Omega)$ be non-negative and subharmonic, i.e. $\Delta u \geq 0$. Use the chain rule to show that the functions $|u|^p$, $p \geq 2$, are subharmonic. What fails for $|u|^p$, $p \in [1, 2)$?

Exercise 7

Assume that $u \in C^2(\Omega)$ is harmonic in Ω.

(a) (1 point) Show that $|u|^p$ is harmonic for some $p \geq 2$ if and only if u is constant?

(b) (2 points) Use the mean value property to show that $\partial_i u$ is (mean-value) harmonic in Ω.

Exercise 8

Let $u \in C^0(\Omega)$ be non-negative and mean-value subharmonic in Ω, i.e. for all $B_r(x) \subset\subset \Omega$ it holds

$$u(x) \leq \fint_{B_r(x)} u(y)dy.$$

(a) (2 points) Use Hölder inequality to show that the functions $|u|^p$, $p \geq 1$, are mean-value subharmonic in Ω. (Hint: Look at the functions $\chi_{B_r(x)} u$ and $\chi_{B_r(x)}$)

(b) (1 point *) Why can the assumption $u \geq 0$ be dropped if u is mean-value harmonic?

(c) (2 points *) Give an alternative proof using Jensen's inequality. Use this to show that $\psi(u)$ is mean-value subharmonic for any strictly increasing, convex function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$.

(d) (1 point) Conclude that for harmonic functions $u \in C^2(\Omega)$, the gradient $|\nabla u|^2(x) := \sum_{i=1}^n (\partial_i u(x))^2$ is mean-value subharmonic.

Exercise 9

(3 points) Let $a^{ij}, b^k : \Omega \to \mathbb{R}$, $i, j, k \in \{1, \ldots, n\}$, be continuous functions on Ω such that $(a^{ij}(x))_{i,j=1}^n \in \mathbb{R}^{n\times n}_{\text{sym}}$ and the operator $L : C^2(\Omega) \to C^0(\Omega)$ defined by

$$Lu(x) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_{ij}u(x) + \sum_{k=1}^{n} b^k(x)\partial_k u(x)$$

is elliptic (with $c = 0$).

(a) Show that if $B \in \mathbb{R}^{n\times n}_{\text{sym}}$ has non-negative spectrum, i.e. $\sigma(B) \subset [0, \infty)$, then the trace $\mathsf{tr}(B) = \sum_{i=1}^n b^{ij}$ is non-negative. (Hint: Use $\mathsf{tr}(B) = \sum_{i=1}^n \langle v_i, Bv_i \rangle$ for any orthonormal basis $\{v_i\}_{i=1}^n$).

(b) Show that any function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Lu(x) > 0$ for $x \in \Omega$ attains its maximum on the boundary of Ω, i.e. $u(x) < \sup_\Omega u$ for all $x \in \Omega$.
(Hint: Show that $\sum a^{ij}(x)\partial_{ij}u(x) = \mathsf{tr}(A \cdot D^2 u)$ where $A = (a^{i,j})_{ij=1}^n$ and $D^2 u$ is the Hessian of u and use properties of the first and second derivatives at (local) maxima.)

Deadline 08.05.2016 in before the lecture.

You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.