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Linear PDE

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Exercise sheet 3

Exercise 10

(2 points) Let $u \in C^0(\Omega)$ be harmonic and $\lambda : [0,\infty) \to [0,\infty)$ be a function with compact support in $[0,\epsilon]$ and $\int_{B_\epsilon(0)} \lambda(\|y\|)dy = 1$. Show that for any $x \in \Omega$ with $d(x,\partial\Omega) > \epsilon$ it holds

$$u(x) = \int_{B_\epsilon(x)} \lambda(\|x-y\|)u(y)dy.$$

(Hint: Use polar coordinates at x!).
(1 points *) Show the corresponding result for mean-value subharmonic functions.

Exercise 11

(2 points) Assume for a given $f \in C^0(\Omega)$ there is a $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\Delta w = f$. Show that for any $g \in C^0(\bar{\Omega})$ the Poisson equation

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u\big|_{\partial\Omega} = g \end{cases}$$

has a solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$.
(1 point *) Using this argument, what is needed to show that the solution is in $C^k(\Omega)$ (resp. in $C^k(\Omega) \cap C^l(\bar{\Omega})$)?

Exercise 12

(3 points) Assume $u \in C^0(\bar{\Omega})$ is non-negative with $\inf_\Omega u = 0$ and satisfies the Harnack inequality, i.e. for all $\Omega' \subset\subset \Omega$ there is a $C > 0$ such that

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u.$$

Show that u satisfies the strong minimum principle, i.e. it holds

$$\inf_\Omega u = \inf_{\partial\Omega} u$$

and if $u(x_0) = \inf_\Omega u = 0$ for some $x_0 \in \Omega$ then u is constant in Ω.

Exercise 13

(Running exercise) Let $(X, \|\cdot\|)$ be a Banach space, i.e. a complete normed vector space, such that the norm $\|\cdot\|$ is p-uniformly convex for some $p \ge 2$, i.e. for there is a constant $C_p > 0$ such that all $v, w \in X$ it holds

$$\left\|\frac{v+w}{2}\right\|^p + C_p \|v-w\|^p \le \frac{1}{2}\|w\|^p + \frac{1}{2}\|v\|^p.$$

Let $C \subset X$ be a closed, bounded and convex[1] set. Define a function $r_C : X \to [0,\infty)$ by

$$r_C(v) := \inf\{\|v-w\| \,|\, w \in C\}$$

[1] C is convex if for all $v, w \in C$ and $\lambda \in [0,1]$ it holds $\lambda v + (1-\lambda)w \in C$

(a) (1 point) Show that the set

$$\pi_C(v) := \{\tilde{w} \in C \mid \|v - \tilde{w}\| = r_C(v)\}$$

contains at most one element. (Hint: $\|w_n - w_m\| = \|(v - w_n) - (v - w_m)\|$.)

(b) (2 points) Fix $v \in X$ and let $(w_n)_{n\in\mathbb{N}}$ be a sequence in C such that

$$r_C(v) = \lim_{n\to\infty} \|v - w_n\|.$$

Show that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. (Hint: Note that $r_C(v) \leq \left\|\frac{(v-w_n)+(v-w_m)}{2}\right\|$.)
Infer that for each $v \in C$ the set $\pi_C(v)$ contains exactly one element which is called the nearest point projection of v onto C.

(c) (2 points *) Since $\pi_C : X \to 2^C$ is single-valued, there is a uniquely defined map $p_C : X \to X$ such that $p_C(v) \in \pi_V(v)$. Show that p_C is continuous.
(Hint: Show r_C is continuous. Use this to show that $(w_n)_{n\in\mathbb{N}} = (p_C(v_n))_{n\in\mathbb{N}}$ is Cauchy if $v_n \to v$.)

Deadline 15.05.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.