EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Faculty of Science

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Linear PDE

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Exercise sheet 4

Exercise 14
(3 points) Let $u \in C^0(\mathbb{R}^n)$ be harmonic and non-constant. Show that u is unbounded, i.e. there are sequences $(x_n^\pm)_{n\in\mathbb{N}}$ such that $u(x_n^\pm) \to \pm\infty$. (Hint: Use Harnack Inequality for Euclidean balls.)

Exercise 15
(2 points) By defininition, a closed set $C \subset \mathbb{R}^n$ is convex if and only if there are linear functions $f_i : \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda_i \in \mathbb{R}$, $i \in I$, such that

$$C = \bigcap_{i\in I}\{f_i \le \lambda_i\}.$$

Show that any bounded convex domain Ω has regular boundary. Note $\bar{\Omega}$ is also convex.

Exercise 16
(1 point) Assume that there is a (mean-value) subharmonic function $b \in C^0(\bar{\Omega})$ such that $b(x_0) > b(x)$ for all $x \in \bar{\Omega}\backslash\{x_0\}$. Show that $x_0 \in \partial\Omega$ is regular.

Exercise 17
(2 points) Assume $b \in C^2(\bar{\Omega} \cap B_r(x_0))$ is a subharmonic function satisfying $b(x) < b(x_0) = 0$ for $x \in \bar{\Omega}\cap B_r(x_0)\backslash\{x_0\}$. Show that it is possible to extend b to a lower barrier function $\tilde{b} : C^0(\bar{\Omega}) \to (-\infty, 0]$ at x_0.

Exercise 18
(Running exercise) Let $(X, \|\cdot\|)$ be a Banach space, i.e. a complete normed vector space, such that the norm $\|\cdot\|$ is p-uniformly convex for some $p \ge 2$, i.e. for there is a constant $C_p > 0$ such that all $v, w \in X$ it holds

$$\left\|\frac{v+w}{2}\right\|^p + C_p\|v-w\|^p \le \frac{1}{2}\|w\|^p + \frac{1}{2}\|v\|^p.$$

(a) (2 points) Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of bounded, closed and convex subsets of X with $C_n \supset C_{n+1}$. Show that

$$\bigcap C_n \ne \varnothing.$$

(Hint: Let $r = \sup_{n\in\mathbb{N}} r_{C_n}$ where $r_{C_n} : X \to [0,\infty)$ is as in the previous sheet. Show that any sequence $w_n \in C_n$ with $r(v) = \lim_{n\in\mathbb{N}}\|v - w_n\|$ is Cauchy and its limit point is in $\cap_{n\in\mathbb{N}}C_n$.)
(1 point *) Why is boundedness of C_n important? Give a counterexample in $\mathbb{R}^n$.

(b) (3 point *) Let $(C_i)_{i\in I}$ be a net of bounded, closed and convex subsets of X with $C_i \supset C_j$ whenever $j \ge i$. Show that $\bigcap_{i\in I} C_i \ne \varnothing$.

(Hint: Choose $w_i \in C_i$ with $\lim_{i\in I}\|v - w_i\| = \sup_{i\in I} r_{C_i}(v)$ and argue as before. Show that the set $I_{i_0} := \{i \in I \mid i \ge i_0\}$ for $i_0 \in I$ is a directed set and the net $(w_i)_{i\in I_{i_0}}$ converges to the same limit as $(w_i)_{i\in I}$. Conclude that w is in every *closed* set C_i.)

Deadline 22.05.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.