EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Faculty of Science

Department of Mathematics

Dr. Martin Kell
Jason Ledwidge

Linear PDE

Summer semester 2017 **15.05.2017**

Exercise sheet 5

Exercise 19

Let Ω be an open, bounded and connected domain in $\mathbb{R}^n$. Assume L is an elliptic operator with $c \leq 0$ and $\sup_{k=1}^n \frac{|b^k|}{\lambda} \leq \mathsf{const}$ where $\lambda : \Omega \to (0, \infty)$ is the lower ellipticity constant of $(a^{i,j} : \Omega \to \mathbb{R})_{i,j=1}^n$. Assume $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and define $u^+ = \max\{0, u\}$.

(a) (2 points) Show the following form of the maximum principle: If $Lu \geq 0$ in Ω then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+.$$

(Hint: Look at $\Omega^+ = \operatorname{int}\{u \geq 0\}$ and use facts of the elliptic operator $\tilde{L} = L - c$.).

(b) (1 point) Conclude that $Lu = 0$ on Ω and $u\big|_{\partial\Omega} = 0$ implies $u = 0$. (Hint: What is the corresponding minimum principle?)

(c) (1 point) Show that if $Lu \geq 0$ and u assumes a *positive* maximum at some $x \in \Omega$ then u is constant in Ω.

Exercise 20

Let Ω be as above. For $T > 0$ set $Q = \Omega \times (0, T)$ and define $\partial' Q = \partial\Omega \times (0, T) \cup \Omega \times \{0\}$. Assume L is an elliptic operator on Ω with $c = 0$ and $u \in C^2(Q) \cap C^0(\bar{Q})$.

(a) (2 points) Show that if $\partial_t u - Lu < 0$ then the strong (parabolic) maximum principle holds in Q, i.e.

$$\sup_Q u = \sup_{\partial' Q} u > u(x, t) \qquad \text{for all } (x, t) \in Q.$$

(Hint: Choose $Q' = \Omega \times (0, T')$ for $T' \in (0, T)$ and argue as in the elliptic case with Q'. The same argument also excludes maximum points in $\Omega \times \{T'\}$. Conclude by letting $T' \to T$).

(b) (1 point) Use the function $v(x, t) := -t$ to show that the weak maximum principle holds for functions u satisfying $\partial_t u - Lu \leq 0$ in Q, i.e. $\sup_Q u = \sup_{\partial' Q} u$.

(c) (1 point) Conclude that $\partial_t u - Lu = \partial_t v - Lv$ and $u\big|_{\partial' Q} = v\big|_{\partial' Q}$ for functions $u, v \in C^2(Q) \cap C^0(\bar{Q})$ implies $u = v$ in Q.

(d) (4 points *) Assume $c = b^k = 0$, $k = 1, \ldots, n$, to show the strong parabolic maximum principle for the parabolic operator $\partial_t - L$. Use functions of the form

$$v_{\alpha,y,t,R}(t, x) = e^{-\alpha\left(\|x-y\|^2 + |t-s|^2\right)} - e^{-\alpha R^2}.$$

Exercise 21

(2 points) Let $u \in C^1((0, 1))$ show for $x, y \in (0, 1)$ and $p \in (1, \infty)$ it holds

$$|u(x) - u(y)| \leq |x - y|^{1 - \frac{1}{p}} \|u'\|_p.$$

Deadline 29.05.2016 in before the lecture.

You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.