EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Faculty of Science

Department of Mathematics

Dr. Martin Kell
Jason Ledwidge

Linear PDE

Summer semester 2017 **29.05.2016**

Exercise sheet 6

Exercise 22
(3 points) Let $p \in [1, n)$. Show that the following inequality

$$\|u\|_r \leq C\|Du\|_p$$

holds for all $u \in C_c^1(\mathbb{R}^n)$ from some C (not depending on u) if and only if $r = p^* = \frac{np}{n-p}$.
(Hint: For one direction, check whether note that $\operatorname{supp} u \subset\subset \Omega$ for some Ω. For the other direction, use chain rule and change of variable applied to $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$ and let $\lambda \to 0$ and resp. $\lambda \to \infty$).

Exercise 23
(3 points) Let $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, x \cdot y > 0, \|(x, y)\| < 1\} = \Omega^+ \dot{\cup} \Omega^- \dot{\cup} \{(0, 0)\}$. Show that for any two constants $c_1, c_2 \in \mathbb{R}$ the function

$$u(x, y) = \begin{cases} c_1 & x > 0 \\ c_2 & x < 0 \end{cases}$$

is in $W^{1,p}(\Omega)$ for $p \in [1, 2)$. (Hint: Construct functions $u_\epsilon \in C^1(\Omega)$ which are constant on $\Omega \backslash B_\epsilon(0, 0)$ and whose partial derivatives are bounded by $\frac{C}{\epsilon}$.)
Remark: The goal of the exercise is to show that u is a $W^{1,p}$-limit of C^1-functions which are continuous and differentiable in the origin $(0, 0)$. For that reason $(0, 0)$ is assumed to be a point of Ω.

Reminder: The optional project is 12.06.2017 as well.

Deadline 12.06.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.