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# Linear PDE

**Summer semester 2017** **18.06.2016**

## Exercise sheet 8

**Exercise 27**

(2 points) Let $T:(X,\|\cdot\|_1)\to(Y,\|\cdot\|_2)$ be a bounded linear map between two Banach spaces such that for some $c>0$ it holds

$$c\|x\|_1\le\|Tx\|_2$$

for all $x\in X$. Show that $T(X)$ is a closed subspace of $Y$. Furthermore, if $Y$ is a Hilbert space then $T$ is onto if and only if $T(X)^\perp=\{\mathbf{0}\}$.

**Exercise 28**

Let $(H,\langle\cdot,\rangle)$ be a Hilbert space and $\alpha:H\to\mathbb{R}$ be a bounded linear map with $\alpha(v)\ne0$ for some $v\in H$.

(a) (2 points) For all closed subspaces $X\le H$ and $w\in H$ there are unique $w_1\in X$ and $w_2\in X^\perp$ such that

$$w=w_1+w_2.$$

(Hint: Show that the nearest-point projection $p_X:H\to X$ is well-defined and $w-p_X(w)\in X^\perp$.)

(b) (1 point) Show that $H_\alpha=(\alpha^{-1}(0))^\perp$ is one-dimensional
(Note: A space $X$ is *not* one-dimensional if there are $v,w\in X$ such that for all $t,s\ne0$ it holds $tv+sw\ne\mathbf{0}$).

(c) (2 point) Conclude that for a unique $v_0\in H$ it holds

$$\alpha(w)=\langle v_0,w\rangle\quad\text{for all } w\in H.$$

(Hint: First show the claim for $w\in H_\alpha$, then use part (a)).

**Exercise 29**

(3 points) Assume $L$ is an elliptic operator on $\Omega$, and $b^k:\Omega\to\mathbb{R}$, $k=1,\dots,n$, and $c:\Omega\to\mathbb{R}$ are bounded. Show that for some $C>0$ the bilinear function $B:W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)\to\mathbb{R}$ defined by

$$B(u,v)=C\cdot B_0(u,v)+\int\sum_{k=1}^n b^k\partial_k u\cdot v+c\cdot u\cdot v dx$$

is bilinear, bounded and coercive where

$$B_0(u,v)=\int\sum_{i=1}^n\partial_i u\cdot\partial_i v dx.$$

Conclude that

$$B_L(u,v)=\int\sum_{i,j=1}^n a^{ij}\partial_i u\cdot\partial_j v+\sum_{k=1}^n b^k\partial_k u\cdot v+c\cdot u\cdot v dx$$

is bilinear, bounded and coercive if the lower ellipticity constant $\lambda$ of $L$ satisfies $\lambda\ge C$.
(Hint: Use Cauchy–Schwarz and Gagliardo–Nirenberg to estimate the integrals of $b^k\partial_k u\cdot v$ and $c\cdot u\cdot v$ from below.)

**Deadline 25.06.2016 in before the lecture.**
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.