EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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Linear PDE

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Exercise sheet 9

Exercise 30

(4 points) Assume Ω is a bounded, connected and open domain in $\mathbb{R}^n$ and $u \in L^p(\Omega)$ with $\operatorname{supp} u \subset\subset \Omega$. Show that $u \in W^{k,p}(\Omega)$ if and only if for all multi-indices I with $|I| = k$, u has weak derivatives $g_I \in L^p(\Omega)$. (Hint: Use mollification and Gagliardo–Nirenberg.)

Exercise 31

(2 points) Let $u \in W^{1,p}(\Omega)$ and $\Omega' \subset\subset \Omega$ be a connected subdomain. Then for all $0 < h < d(\Omega', \partial\Omega)$ it holds $\|\partial_i^h u\|_{L^p(\Omega')} \leq \|\partial_i u\|_{L^p(\Omega)}$ where

$$\partial_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

Exercise 32

(2 points) Observe the following fact: If $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega)$ then there is a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and an element $w \in L^p(\Omega)$ such that for all bounded linear functions $\alpha : L^p(\Omega) \to \mathbb{R}$ it holds $\lim_{k\to\infty} \alpha(w_{n_k}) = \alpha(w)$.
Assume $u \in L^p(\Omega)$ and there is a constant K such that for all connected $\Omega' \subset\subset \Omega$, $0 < h < d(\Omega', \partial\Omega)$ and $i \in \{1, \dots, n\}$ it holds $\|\partial_i^h u\|_{L^p(\Omega')} \leq K$. Use the observation to show that u has weak derivatives $g_i \in L^p(\Omega)$ with $\|g_i\|_{L^p(\Omega)} \leq K$. Conclude $u \in W^{1,p}(\Omega)$.

Exercise 33

(Running exercise) Let $(X, \|\cdot\|)$ be a Banach space with r-uniformly convex norm. We define the co-convex topology τ_{co} on X as the smallest topology such that every closed convex set $C \subset X$ is τ-closed, i.e.

$$\{U \subset X \,|\, X\backslash U \text{ is closed and convex}\}$$

is a subbase for τ.

(a) (2 point) If $U_i \subset X$, $i \in \mathbb{N}$, is such that $X\backslash U_i$ is convex and $\cup_{i\in\mathbb{N}} U_i \supset C$ for a bounded, closed and convex set C then there is a finite subset $J \subset \mathbb{N}$ such that $\cup_{i\in J} U_i \supset C$.

(b) (3 points *) If $U_i \subset X$, $i \in I$, is such that $X\backslash U_i$ is convex and $C \subset \cup_{i\in I} U_i$ for a bounded, closed and convex set $C \subset X$ then there is a finite subset $J \subset I$ such that $C \subset \cup_{i\in J} U_i$. (Hint: Use (transfinite) induction and show that it ends after finitely many steps.)
Remark: By the Alexander Subbase Theorem this implies that every closed bounded and convex set $C \subset X$ is τ-compact. In particular, any bounded sequence in X admits a τ-convergent subsequence.

(c) (2 points *) If $f : X \to \mathbb{R} \cup \{\infty\}$ is convex and lower semi-continuous w.r.t. norm topology then f is lower semi-continous w.r.t. τ_{co}. In particular, any bounded linear functional $\alpha : X \to \mathbb{R}$ is continous w.r.t. τ.

Deadline 03.07.2016 in before the lecture.
You reach the website of the lecture under `https://tinyurl.com/UniTue-LinPDE`.