Optimal transport and non-branching geodesics

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L^p-Monge problem

- Solve for *good* μ and arbitrary ν the following

$$\inf_{\nu=T_*\mu}\int d^p(x,T(x))d\mu(x)$$

- When is the solution unique?

L^p-Monge-Kantorovich problem

- Show that the minimum of

$$\inf_{\pi\in\Pi(\mu,\nu)}\int d^p(x,y)d\pi(x,y)$$

is supported on a graph of measurable map.

- For $p = 1$ almost never true.
- For $p \in (1,\infty)$ depends on the geometry and on μ.
- If true then
  - the optimal coupling is unique
  - Monge = Kantorovich.

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geodesics

BLACKBOARD

Non-branching geodesics

- Assumption: $(M, d, \mathbf{m})$ a complete geodesic measure space

Definition (non-branching)

A geodesic space (M, d) is non-branching if for all geodesics $\gamma, \eta : [0,1] \to M$ with $\gamma_0 = \eta_0$ and $\gamma_t = \eta_t$ for some $t \in (0,1)$ it holds $\gamma_t = \eta_t$ for all $t \in [0,1]$.

Equivalently:
If m is a midpoint of (x, y) and (x, z) then $y = z$.

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Non-branching and optimal transport

Lemma (no intermediate overlap)

A geodesic space is non-branching if the following holds: Whenever for two geodesics γ and η satisfy

$$d^p(\gamma_0, \gamma_1) + d^p(\eta_0, \eta_1) \leq d^p(\gamma_0, \eta_1) + d^p(\eta_0, \gamma_1)$$

and $\gamma_t = \eta_t$ for some $t \in (0, 1)$ then $\gamma \equiv \eta$.

Examples of non-branching space

- Riemannian/Finsler manifolds (geodesic = "ODE solution")
- Alexandrov spaces (comparison condition)
- Busemann G-spaces (unique continuation property)
- $\mathrm{CAT}(\kappa) \oplus \mathrm{RCD}(\mathrm{K}, \mathrm{N})$-space [Kapovich-Ketterer '17]
  $\Longrightarrow$ works also for $\mathrm{MCP}_{loc}(K, N)$-spaces that are (locally) Busemann convex
- subRiemannian Heisenberg(-type) groups [Ambrosio-Rigot '04]
- subRiemannian Engel group [Ardentov-Sachkov '11,'15]
- Open: Ricci limits, RCD-spaces, Carnot groups

Some history of L^p-Monge-Kantorovich, $p > 1$

- Theorems using Rademacher Theorem
  - in $\mathbb{R}^n$ [Brenier '91, Gangbo-McCann '96]
  - Riemannian manifolds [McCann '01, Gigli '11]
  - Finsler manifolds [Villani '09, Ohta '09]
  - Heisenberg groups [Ambrosio-Rigot '04]
  - nice subRiemannian manifolds [Figalli-Riffort '10]
  - Alexandrov spaces [Bertrand '08/'15, Schultz-Rajala '18]
- Anyone missing?

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- Theorems using optimal transport and non-branching
  - non-branching $\mathrm{CD}(K, N)$-spaces [Gigli '12]
  - strongly non-branching doubling spaces with interpolation property [Ambrosio-Rajala '14]
  - non-branching spaces with very weak MCP [Cavalletti-Huesmann '15]
- using weaker essentially non-branching (e.n.b.) condition
  - strong $\mathrm{CD}(K, \infty)$-spaces [Rajala-Sturm '14]
  - $\mathrm{RCD}(K, N)$-spaces [Gigli-Rajala-Sturm '16]
  - e.n.b. $\mathrm{MCP}(K, N)$-spaces [Cavalletti-Mondino '17]
  - e.n.b. spaces with very weak MCP [K. '17]

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Notation

- $\Gamma \subset M \times M$

$$\Gamma_t := \{\gamma_t \,|\, \gamma \in \mathrm{Geo}_{[0,1]}(M,d), (\gamma_0, \gamma_1) \in \Gamma\}$$

- $A \subset M$ and $x \in M$

$$A_{t,x} = (A \times \{x\})_t$$

Remark

In the following fix a $p \in (1, \infty)$ so that optimal $=$ d^p-optimal, cylically monotone $=$ d^p-cylically monotone.

Very weak MCP condition

Definition ([Cavalletti-Huesmann '15])

A metric measure space is qualitatively non-degenerate if for all $R > 0$ there is a function $f_R : (0,1) \to (0,\infty)$ with $C_R = \limsup_{t\to 0} f_R(t) > \frac{1}{2}$ such that whenever $\{x\}, A \subset B_R(x_0)$ then

$$\mathbf{m}(A_{t,x}) \geq f_R(t)\mathbf{m}(A).$$

Remark

Note that $2C_R > 1$.

Optimal transport maps

Definition (Good Transport Behavior)

A metric measure space (M, d, m) has good transport behavior $(GTB)_p$ if for all $\mu \in \mathcal{P}_p^{ac}(M)$ and all $\nu \in \mathcal{P}_p(M)$ every optimal coupling π is induced by a transport map T, i.e. $\pi = (\mathrm{id} \times T)_* \mu$.

Theorem ([Cavalletti-Huesmann '15])

Assume (M, d, m) is qualitatively non-degenerate and non-branching. Then (M, d, m) has good transport behavior $(GTB)_p$ for all $p \in (1, \infty)$.

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Proof for $\nu = (1-\lambda)\delta_{x_1} + \lambda\delta_{x_2}$, $x_1 \neq x_2$

- Choose some optimal coupling π and note

$$\operatorname{supp}\pi = A_1 \times \{x_1\} \dot\cup A_2 \times \{x_2\} \dot\cup A \times \{x_1, x_2\}.$$

- Observation:
  - π is induced by a transport map iff $\mathbf{m}(A) = 0$.
  - by non-branching for $t \in (0,1)$

$$A_{t,x_1} \cap A_{t,x_2} = \varnothing.$$

  - by qualitative non-degeneracy (and A is compact)

$$\begin{aligned}\mathbf{m}(A) &\geq \limsup_{t\to 0} \mathbf{m}(A_{t,x_1} \cup A_{t,x_2}) \\ &= \limsup_{t\to 0} \mathbf{m}(A_{t,x_1}) + \mathbf{m}(A_{t,x_2}) \\ &= 2\limsup_{t\to 0} f(t)\mathbf{m}(A) = 2C_R\mathbf{m}(A).\end{aligned}$$

- Conclusion: $\mathbf{m}(A) = 0$ and π is induced by a transport map.

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Proof for $\nu = \sum_{i=1}^n \lambda_i \delta_{x_i}$

- Choose some optimal coupling π then

$$\operatorname{supp} \pi = \bigcup_{i=1}^n A_i \times \{x_i\} \dot\cup \bigcup_{i<j}^n A_{ij} \times \{x_i, x_j\}.$$

- By previous slide $\mathbf{m}(A_{ij}) = 0$.
- Hence

$$T(x) = \begin{cases} x_i & x \in A_i \\ x & \text{otherwise} \end{cases}$$

is a transport map.

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Observation

- For distinct $x, y \in M$ and compact $A \subset M$

$$A_= = \{z \in M \,|\, d(x,z) = d(y,z)\}$$
$$A_{\neq} = A \backslash A_=.$$

- If $A \times \{x, y\}$ is cyclically monotone then even without non-branching

$$(A_{\neq})_{t,x} \cap (A_{\neq})_{t,y} = \varnothing.$$

- Hence if

$$\forall x \neq y : \mathbf{m}(\{z \in M \,|\, d(x,z) = d(y,z)\} = 0$$

then any optimal coupling between $\mu \ll \mathbf{m}$ and ν discrete is induced by a transport map.

- This holds for **any** normed space and measure assigning zero mass to hyperplanes.

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Proof for general ν

Theorem (Selection dichotomy, see e.g. [K. '17])

*If some optimal coupling π is **not** induced by a transport map then there is a compact set $K \subset \mathrm{supp}(p_1)_*\pi$ with $\pi(K \times M) > 0$ and two continuous maps $T_1, T_2 : M \to M$ with $T_1(K) \cap T_2(K) = \varnothing$ such that*

$$\Gamma^{(1)} \cup \Gamma^{(2)}$$

is cyclically monotone where $\Gamma^{(i)} = \mathrm{graph}_K T_i$.

Lemma

If (M,d) is non-branching then

$$\Gamma_t^{(1)} \cap \Gamma_t^{(2)} = \varnothing$$

and

$$\mathbf{m}(K) \geq \limsup_{t \to 0} \left[\mathbf{m}(\Gamma_t^{(1)}) + \mathbf{m}(\Gamma_t^{(2)}) \right]$$

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Uniform approximation (simplified for presentation)

Lemma

If $(M, d, \mathbf{m})$ *is non-branching and qualitatively non-degenerate then*

$$\mathbf{m}(\Gamma_t^{(i)}) \geq f(t)\mathbf{m}(K).$$

Idea of proof.

Let $\nu_n \to (T_i)_*\mu\big|_K$ with ν_n discrete then eventually

$$\Gamma_t^{(i),n} \overset{!}{\subset} (\Gamma_t^{(i)})_\epsilon$$

so that

$$\mathbf{m}(\Gamma_t^{(i)}) \geq \limsup_{n\to 0} \mathbf{m}(\Gamma_t^{(i),n}) \geq f(t)\mathbf{m}(K).$$

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Idea of proof.

Let $\nu_n \to (T_i)_* \mu\big|_K$ with ν_n discrete then eventually

$$\Gamma_t^{(i),n} \overset{!}{\subset} (\Gamma_t^{(i)})_\epsilon$$

so that

$$\mathbf{m}(\Gamma_t^{(i)}) \geq \limsup_{n\to 0} \mathbf{m}(\Gamma_t^{(i),n}) \geq f(t)\mathbf{m}(K).$$

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Conclusion

- If the claim was wrong then using we arrive at the following contradiction

$$\begin{aligned}\mathbf{m}(K) &\geq \limsup_{t\to 0} \mathbf{m}(\Gamma_t^{(1)}) + \mathbf{m}(\Gamma_t^{(2)}) \\ &\geq 2\limsup_{t\to 0} f(t)\mathbf{m}(K) = 2C_R\mathbf{m}(K).\end{aligned}$$

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- qualitative non-degeneracy $\Longrightarrow$ not an optimal transport property

Partial extension to $p=1$, see e.g. [K.-Suhr 18]

- For simplicity let $M=\mathbb{R}^n$

Lemma

If $\Gamma\subset\{x_n<0\}\times\{x_n=0\}$ is d-cyclically monotone then for all distinct geodesics γ,η with

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Assume $\operatorname{supp}\mu\times\operatorname{supp}\nu\subset\{x_n<0\}\times\{x_n=0\}$ then any d-optimal coupling is induced by a transport map.

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Works for non-bran., qual. non-deg. spaces if ν is supported in a level set of a dual solution.

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Lorentzian setting [K.-Suhr '18]

- Lagrangian for $p \in (0,1]$

$$\mathcal{L}_p(v) = \begin{cases} -(-g(v,v))^{\frac{p}{2}} & v \in \bar{\mathcal{C}} \\ \infty & \text{otherwise} \end{cases}$$

induces cost function $c_p : M \times M \to (-\infty, 0] \cup \{\infty\}$

- For $p \in (0,1)$, geodesics connecting $(x,y) \in c_p^{-1}((-\infty,0))$ are non-branching.
- For $p = 1$ and hyperbolic spacetimes, introduce *smooth* time function $\tau : M \to \mathbb{R}$ with

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- Assume $(M, d, \mathbf{m})$ has good transport behavior $(GTB)_p$
  - for every $\mu_0 \ll \mathbf{m}$ and μ_1 the optimal coupling is unique and induced by a transport map
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- Let $T = [0,1]_1 \cup [0,1]_3 \cup [0,1]_3$ be the tripod glued at 0.
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Essentially non-branching (I)

- A subset of geodesics $\mathcal{G} \subset \mathrm{Geo}_{[0,1]}(M,d)$ is called non-branching if for all $\gamma, \eta \in \mathcal{G}$ with $\gamma_0 = \eta_0$ or $\gamma_1 = \eta_1$ it holds whenever $\gamma_t = \eta_t$ for some $t \in (0,1)$ it holds $\gamma_t = \eta_t$ for all $t \in [0,1]$.

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Theorem ([K. '17])

If (M,d,m) has good transport behavior then any dynamical coupling σ with $(e_0)_*\sigma \ll \mathrm{m}$ is concentrated on a set that is non-branching to the left. In particular, (M,d,m) is essentially non-branching $(ENB)_p$.

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Obstructions to good transport behavior (II)

- Choose three mutually singular measures $\mathbf{m}_1$,$\mathbf{m}_2$ and $\mathbf{m}_3$ on $[0,1]$.
- For the tripod (T,d), regard them as measures on $[0,1_i]$ and set $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3$
- Observations:
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Theorem ([K. '17])

If $(M, d, \mathbf{m})$ *is essentially non-branching* $(ENB)_p$ *and qualitatively non-degenerate then it has good transport behavior* $(GTB)_p$.

Idea of the proof

- As $\nu \not\ll \mathbf{m}$ essentially non-branching cannot be directly used
- Construct dynamical optimal coupling σ with $(e_\epsilon)_*\sigma, (e_{1-\epsilon})_*\sigma \ll \mathbf{m}$ with

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for a cyclically monotone set Γ with $(e_0, e_{1-\epsilon})_*\sigma(\Gamma) = 1$.

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Intermediate summary

- Assume $(M, d, \mathbf{m})$ essentially non-branching and qualitatively non-degenerate
- Let $\mu_0 = f_0\mathbf{m}$ and μ_1 arbitrary
- Conclusion:
  1. (uniqueness) unique optimal dynamical coupling σ
  2. (good transport behavior)
  $$(e_0, e_1)_*\sigma = (\mathrm{id} \times T_1)_*\mu_0$$
  3. (strong interpolation property)
  $$(e_t)_*\sigma = f_t\mathbf{m}$$
  4. (strong bounded density property)
  $$f_t(\gamma_t) \leq \frac{1}{f_R(t)} f_0(\gamma_0).$$

Application to Measure Rigidity

Theorem ([K. '17])

Assume (M, d, m_1) and (M, d, m_2) are both essentially non-branching $(ENB)_p$ and qualitatively non-degenerate then m_1 and m_2 are mutually absolutely continuous.

Corollary

For $i = 1, 2$ let (M, d, m_i) be $\mathrm{RCD}(K_i, N_i)$-spaces with $N_i < \infty$. Then m_1 and m_2 are mutually absolutely continuous.

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- Decompose $\mathbf{m}_2 = f\mathbf{m}_1 + \mathbf{m}_2^s$
- Assume, by contradiction, $\mathbf{m}_2^s \neq 0$.
- Observation: We must have $f \neq 0$ by strong interpolation property
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- Note if $\mathbf{m}_1(A_{t,x}) = \mathbf{m}_1(A) = 0$ and then

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2nd Proof of the Measure Rigidity

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- By strong intersection property

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hence $\mu_t \perp \mathbf{m}_2^s$

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- Arrive at contradiction using the following lemma.

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If $\mu = \frac{1}{\mathbf{m}(K)} \mathbf{m}\big|_K$ *and* $\mu_n = f_n \mathbf{m}$ *with* $W_p(\mu_n, \mu) \to 0$ *and* $\|f_n\|_\infty \leq C$ *then* $\mu \not\perp \mu_n$ *for all sufficiently large* n.

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Thank you for your attention