EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Sectional curvature-like conditions on metric spaces

Martin Kell

Universität Tübingen

Sanya, March 2017

How to recognize curvature?

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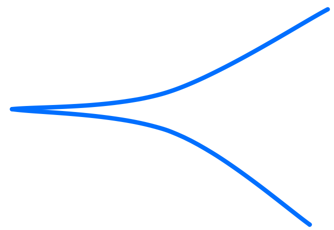
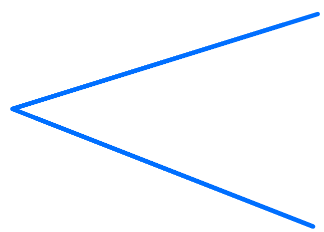
How to recognize curvature?

with geodesics

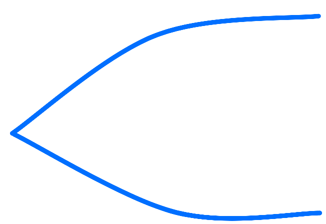
$K < 0$



$K = 0$



$K > 0$



How to recognize curvature?

with geodesics | with volume

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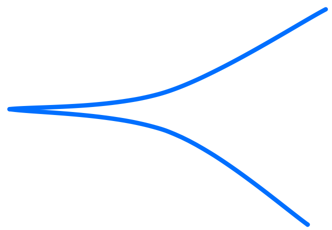

isoperimetric ratio for discs
[Lytchak-Wenger '16]

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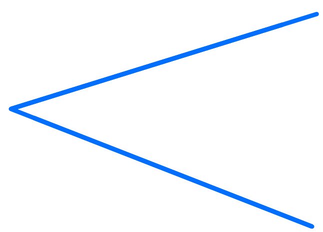
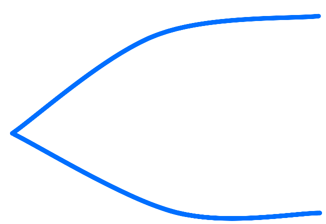


sharp volume growth

$K > 0$



· measure contraction property
[Sturm, Ohta]

How to recognize curvature?

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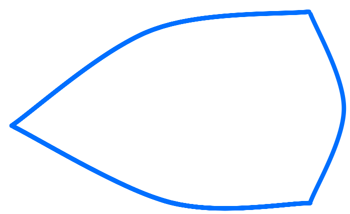
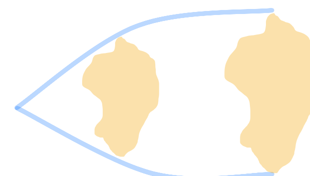
Alex(k)

with volume

MCP(k,N)

$k > 0$





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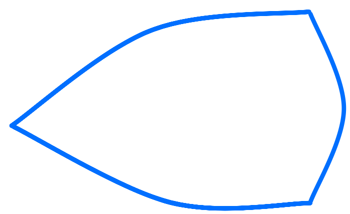
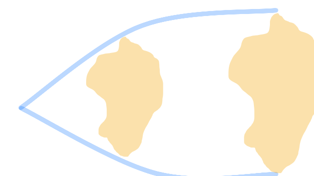
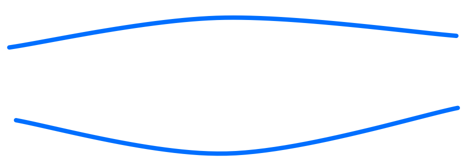




Rauch II [Berger]



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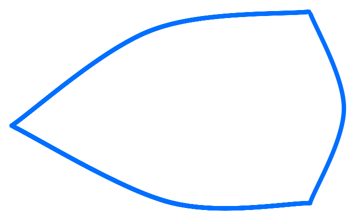

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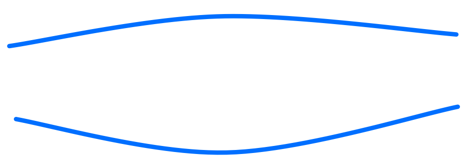
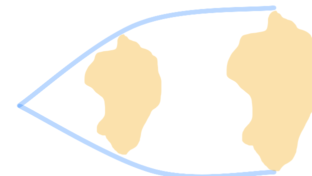
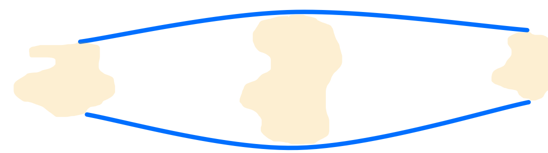


with volume

MCP(k,N)



CD(k,N) [Lott-Villani Sturm]



How to recognize curvature?

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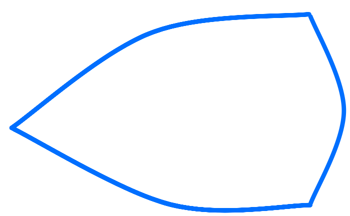

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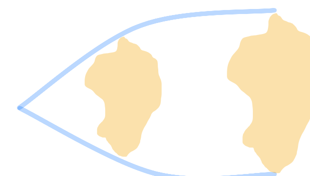

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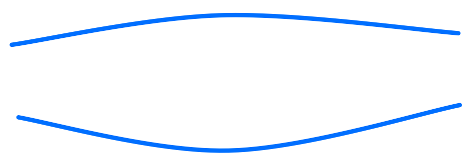
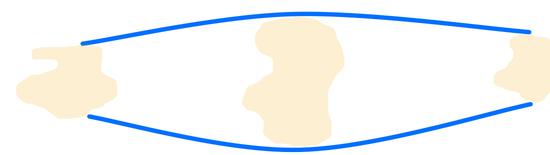


Riemannian!

CD(K,N) [Lott-Villani
Sturm]



also Finsler! [Ohta]

For $k \leq 0$ there are two important properties

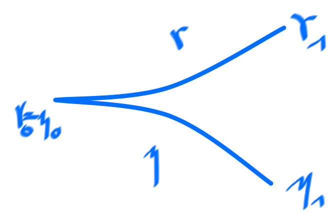
For $K \le 0$ there are two important properties

- CAT(0)-inequality:

$$d^2(x_{1/2}, z) \le \frac{1}{2} d^2(x_0, z) + \frac{1}{2} d^2(x_1, z) - \frac{1}{4} d^2(x_0, x_1)$$

- Busemann convexity

$$d(\gamma_t, \eta_t) \le t\, d(\gamma_0, \eta_0)$$



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- uniform convexity

(additive) $$d^p(x_{\frac{1}{2}}, z) \leq \frac{1}{2} d^p(x_0, z) + \frac{1}{2} d^p(x_1, z) - \frac{C}{4} d^p(x_0, x_1) \qquad p \geq 2$$

(multiplicative) $$d^p(x_{\frac{1}{2}}, z) \leq \left(1 - \delta(\varepsilon)\right)\left(\frac{1}{2} d^p(x_0, z) + \frac{1}{2} d^p(x_1, z)\right) \qquad p \in (1, \infty)$$

with $d(x_0, x_1) \geq \varepsilon \left(\frac{1}{2} d^p(x_0, z) + \frac{1}{2} d^p(x_1, z)\right)$

Known results for uniformly convex geodesic spaces

- unique geodesics
- weak topology ($\hat{=}$ co-convex topology) is boundedly compact
- weak convergence (via asym. centers) is sequentially compact
- Banach-Saks [Yokota '13 $p=2$; K. '15 $p\in(1,\infty)$]

  "roughly" $\sum_{i=1}^{n} \frac{x_i}{n}$ has strongly conv. subsequence if $x_n \to x$ weakly
- If space is Busemann convex then $(UC)_p$ for some $p\in[1,\infty]$ $\Rightarrow$ $(UC)_p$ for all $p\in[1,\infty]$

  [Foertsch '04]

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- uniform smoothness

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Busemann fuction associated to a ray $\gamma : [0,\infty) \to M$

$$b_\gamma(x) = \lim_{t\to\infty} t - d(x, \gamma_t)$$

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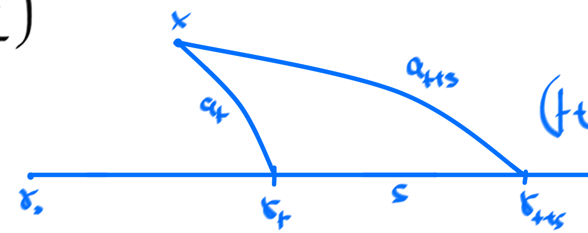

$a_{t+s} \le a_t + s$

$(t+s) - a_{t+s} \ge a_t - t$

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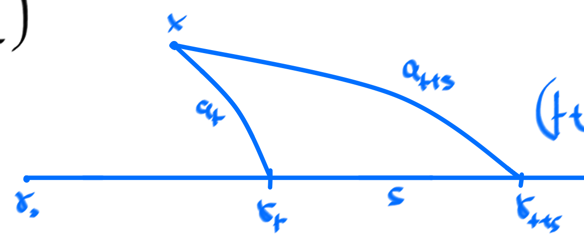

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Theorem: If (M, d) is uniformly smooth then b_γ is convex

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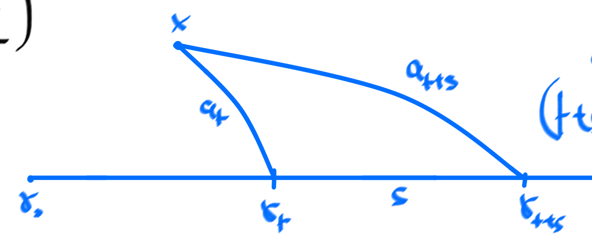

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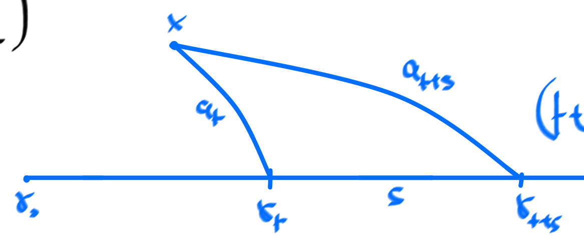

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Theorem: If (M,d) is uniformly smooth then b_γ is convex

Proof:

$$d^p(x_{\frac12}, z) \ge \tfrac12\big(\qquad d^p(x_0,z)\big) + \tfrac12\big(\qquad d^p(x_1,z)\big) - \tfrac{C}{4} d^p(x_0,x_1)$$

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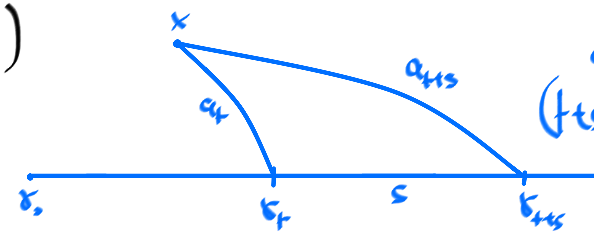

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Theorem: If (M, d) is uniformly smooth then b_γ is convex

Proof:

$$\frac{d^p(x_{\frac{1}{2}}, z)}{t^{p-1}} \geq \frac{1}{2}\left(\frac{d^p(x_0, z)}{t^{p-1}} \right) + \frac{1}{2}\left(\frac{d^p(x_1, z)}{t^{p-1}} \right) - \frac{C}{4} \frac{d^p(x_0, x_1)}{t^{p-1}}$$

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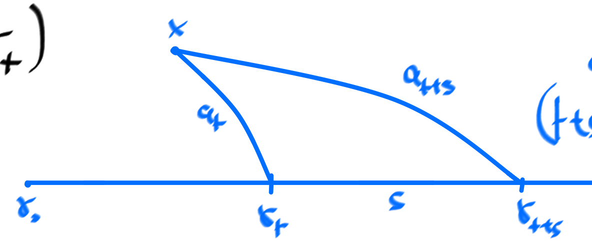

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$$t - \frac{d^p(x_{\frac{1}{2}}, z)}{t^{p-1}} \le \frac{1}{2}\left(t - \frac{d^p(x_0,z)}{t^{p-1}}\right) + \frac{1}{2}\left(t - \frac{d^p(x_1,z)}{t^{p-1}}\right) + \frac{C}{4}\frac{d^p(x_0,x_1)}{t^{p-1}}$$

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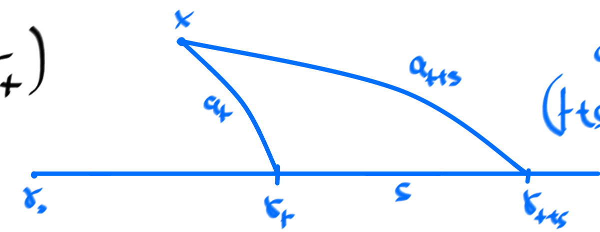

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$$t - \frac{d^p(x_{\frac12}, z)}{t^{p-1}} \le \frac12\left(t - \frac{d^p(x_0,z)}{t^{p-1}}\right) + \frac12\left(t - \frac{d^p(x_1,z)}{t^{p-1}}\right) + \frac{C}{4}\frac{d^p(x_0,x_1)}{t^{p-1}}$$

for $t\to\infty$ $\quad\longrightarrow b_\gamma(x_{\frac12}) \quad\longrightarrow b_\gamma(x_{\frac12}) \quad\longrightarrow b_\gamma(x_{\frac12}) \quad\longrightarrow 0$

$$\Longrightarrow \quad b_\gamma(x_{\frac12}) \le \frac12 b_\gamma(x_0) + \frac12 b_\gamma(x_1)$$

$\square$

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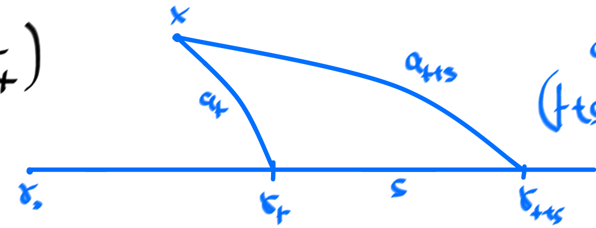

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$$t - \frac{d^p(x_{\frac{1}{2}}, z)}{t^{p-1}} \le \frac{1}{2}\left(t - \frac{d^p(x_0,z)}{t^{p-1}}\right) + \frac{1}{2}\left(t - \frac{d^p(x_1,z)}{t^{p-1}}\right) + \frac{C}{4}\frac{d^p(x_0,x_1)}{t^{p-1}}$$

$$\longrightarrow b_\gamma(x_{\frac{1}{2}}) \qquad \longrightarrow b_\gamma(x_{\frac{1}{2}}) \qquad \longrightarrow b_\gamma(x_{\frac{1}{2}}) \qquad \longrightarrow 0$$

$$\Longrightarrow \quad b_\gamma(x_{\frac{1}{2}}) \le \frac{1}{2} b_\gamma(x_0) + \frac{1}{2} b_\gamma(x_1)$$

$\square$

$\rightsquigarrow$ first step in the Cheeger–Gromol soul construction

Curvature conditions inspired by Busemann

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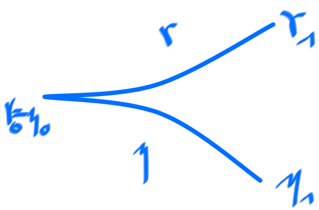
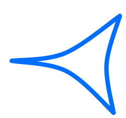
Busemann convexity
[Busemann 1950s]



$$d(\gamma_t, \eta_t) \leq t\, d(\gamma_1, \eta_1)$$

Examples:

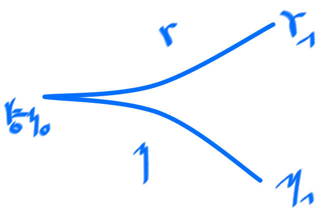
- $\sec \leq 0$
- CAT(0)-spaces



- Berwald spaces $\sec_{\Pi,y} \leq 0$

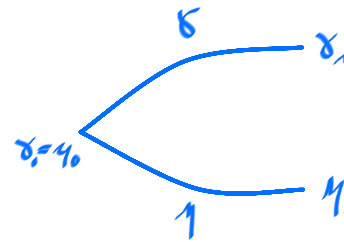
Curvature conditions inspired by Busemann

Busemann convexity [Busemann 1950s]



$$d(\gamma_t, \eta_t) \le t\, d(\gamma_1, \eta_1)$$

Busemann concavity [K. '16] related [Kelly-Strauss '58; Kann '61]



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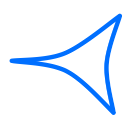
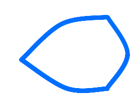
- $\sec \le 0$
- CAT(0)-spaces
- Berwald spaces $\sec_{H_g} \le 0$

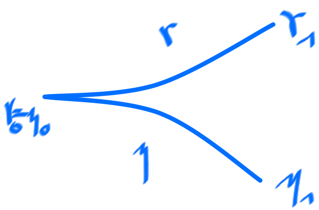


- $\sec \ge 0$
- Alex(0)-spaces
- metric products
- open: higher rank sym. Berwald spaces



Curvature conditions inspired by Busemann

Busemann convexity [Busemann 1950s]



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Examples:

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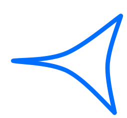
- CAT(0)-spaces
- Berwald spaces $\sec_{fl,y} \leq 0$



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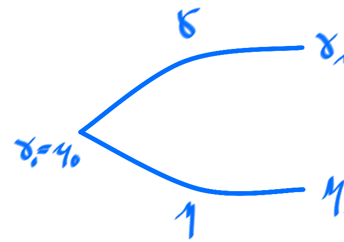

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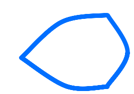


- metric products
- open: higher rank sym. Berwald spaces

Extensions to other bounds via comparison ($K \geq c > 0 \Rightarrow$ Bonnet-Myers)

Infinitesimal to global property

Theorem:

- $\mathrm{curv}_{\mathrm{Buse}} \geq 0$ + tangents are $(US)_{p,c}$
  $\Rightarrow (US)_{p,c}$
- $\mathrm{curv}_{\mathrm{Buse}} \leq 0$ + tangents are $(UC)_{p,c}$
  $\Rightarrow (UC)_{p,c}$

Infinitesimal to global property

Theorem:

- $\mathrm{curv}_{\mathrm{Buse}} \geq 0$ ⊕ tangents are $(US)_{p,c}$ $\Rightarrow$ $(US)_{p,c}$
- $\mathrm{curv}_{\mathrm{Buse}} \leq 0$ ⊕ tangents are $(UC)_{p,c}$ $\Rightarrow$ $(UC)_{p,c}$

Corollary:

- $\mathrm{curv}_{\mathrm{Buse}} \geq 0$ ⊕ tangents are $Alex(0)$ $\Rightarrow$ $Alex(0)$
- $\mathrm{curv}_{\mathrm{Buse}} \leq 0$ ⊕ tangents are $CAT(0)$ $\Rightarrow$ $CAT(0)$

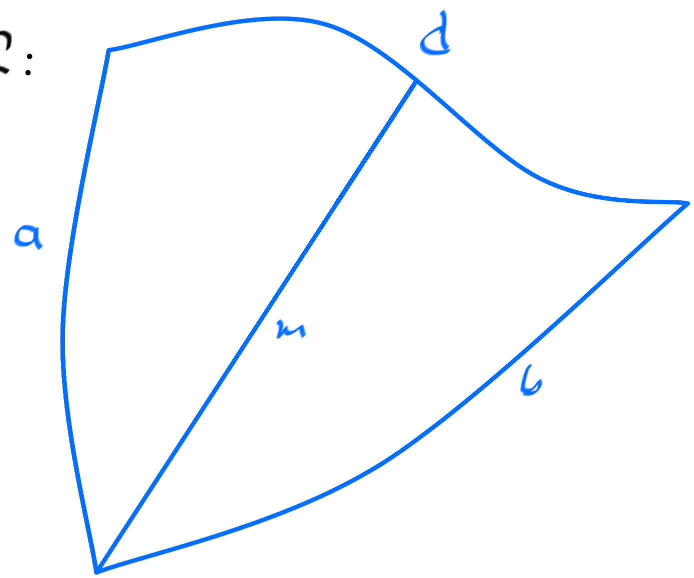
Open: local $\mathrm{curv}_{\mathrm{Buse}} \geq 0 \overset{?}{\Rightarrow} \mathrm{curv}_{\mathrm{Buse}} \geq 0$ (true for $Alex(0)$)

(would imply Berwald spaces with $\sec_{\mathrm{flag}} \geq 0 \Rightarrow \mathrm{curv}_{\mathrm{Buse}} \geq 0$)

Infinitesimal to global property

Theorem: • $\mathrm{curv}_{\mathrm{Buse}} \geq 0$ ⊕ tangents are $(US)_{p,c}$
$\Rightarrow (US)_{p,c}$

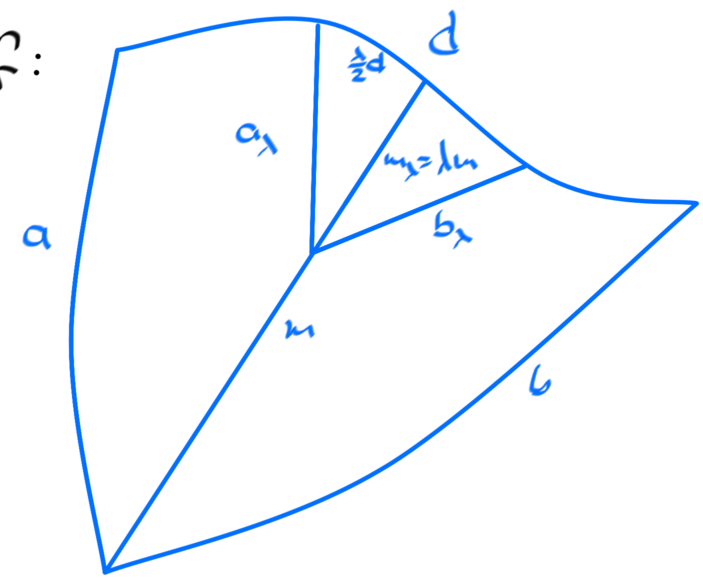
Proof:



Infinitesimal to global property

Theorem: • $\mathrm{curv}_{\mathrm{Buse}} \geq 0$ ⊕ tangents are $(US)_{p,c}$
$\Rightarrow (US)_{p,c}$

Proof:



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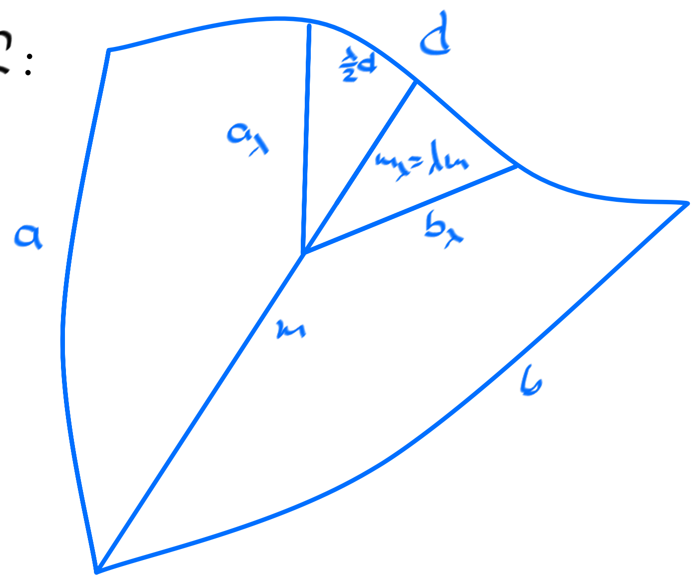

• $\mathrm{curv}_{\mathrm{Busx}} \geq 0$: $\frac{a_\lambda}{\lambda} \geq a$, $\frac{b_\lambda}{\lambda} \geq b$

Infinitesimal to global property

Theorem: • $\mathrm{curv}_{\mathrm{Buse}} \geq 0$ + tangents are $(US)_{p,C}$
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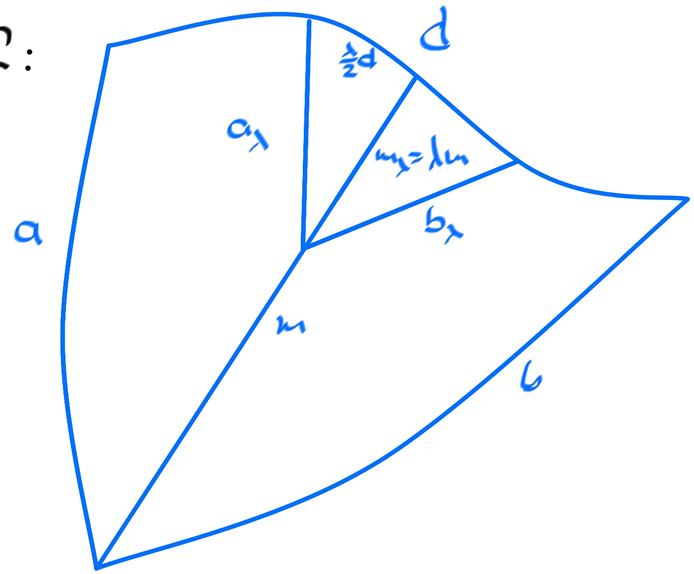
Proof:



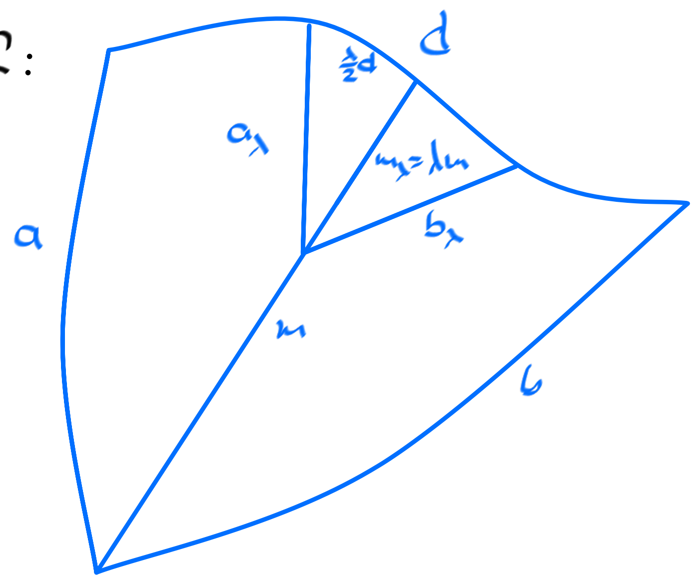
- $\mathrm{curv}_{\mathrm{Buse}} \geq 0$: $\frac{a_\lambda}{\lambda} \geq a$, $\frac{b_\lambda}{\lambda} \geq b$
- $x_0 := \lim_{\lambda \to 0} \frac{x_\lambda}{\lambda}$, $x \in \{a, b, d, m\}$
- tangents are $(US)_{p,C}$
  $m_0^p \geq \frac{1}{2} a_0^p + \frac{1}{2} b_0^p - \frac{C}{4} d_0^p$

Infinitesimal to global property

Theorem: • $\mathrm{curv}_{\mathrm{Buse}} \geq 0 \oplus$ tangents are $(US)_{p,c}$

$\Rightarrow (US)_{p,c}$

Proof:



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  $m_0^p \geq \frac{1}{2} a_0^p + \frac{1}{2} b_0^p - \frac{c}{4} d_0^p$
- $m_0 = m$, $d_0 = d$

$\Rightarrow m^p \geq \frac{1}{2} a^p + \frac{1}{2} b^p - \frac{c}{4} d^p$

$\square$

Measure contraction property

$\Phi_{t,x} : M \to 2^M$, $y \mapsto \{ \gamma_t \mid \gamma$ is geodesic connecting x and $y \}$

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Measure contraction property

$\Phi_{t,x} : M \to 2^M$, $y \mapsto \{ \gamma_t \mid \gamma \text{ is geodesic connecting } x \text{ and } y \}$

- If $\Phi_{t,x}(A) \subset \bigcup_{n \in \mathbb{N}} U_n$ w/ $\operatorname{diam} U_n \leq \varepsilon$
  then $A \subset \bigcup_{n \in \mathbb{N}} \Phi_{t,x}^{-1}(U_n)$ w/ $\operatorname{diam} \Phi_{t,x}^{-1}(U_n) \leq t^{-1} \varepsilon$

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$$\mathcal{H}^N(A) := \lim_{\varepsilon\to 0} \inf\{c_N \sum (\operatorname{diam} U_n)^N : A \subset \bigcup_{n\in\mathbb{N}} U_n, \operatorname{diam} U_n \leq \varepsilon\}$$

$$\leq t^{-N}\mathcal{H}^N(\Phi_{t,x}(A))$$

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$\mathcal{H}^N$ non-trivial $\rightsquigarrow$ measure contraction property $MCP(0,N)$

Bi-Lipschitz Splitting Theorem

Theorem: $\mathrm{curv}_{\mathrm{Buse}} \geq 0 \oplus \exists\, \gamma : \mathbb{R} \to M$ line

$$\Rightarrow (M, d) \xleftrightarrow{\text{bi-Lip}} \left(\tilde{M} \times \mathbb{R},\ \tilde{d} \otimes_{\ell^2} |\cdot - \cdot|\right)$$

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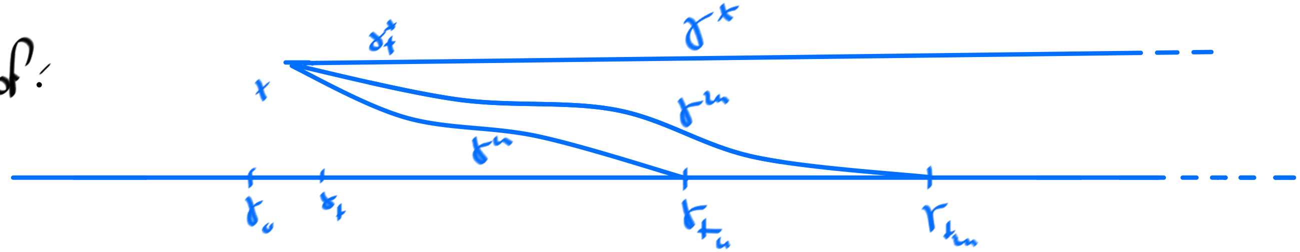
Sketch of Proof:



assume $\gamma^m \to \gamma^x$

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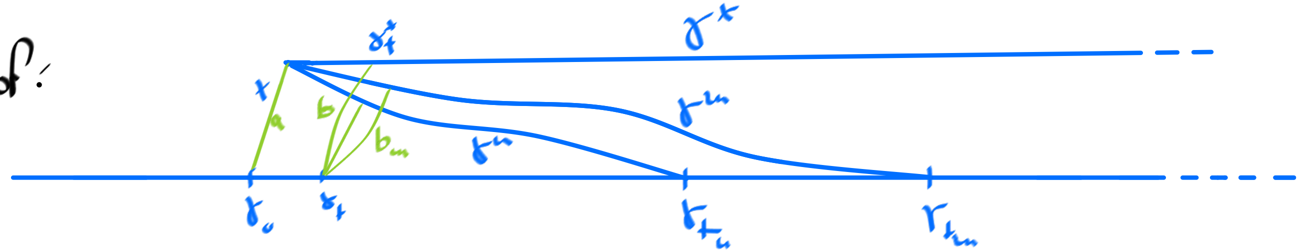

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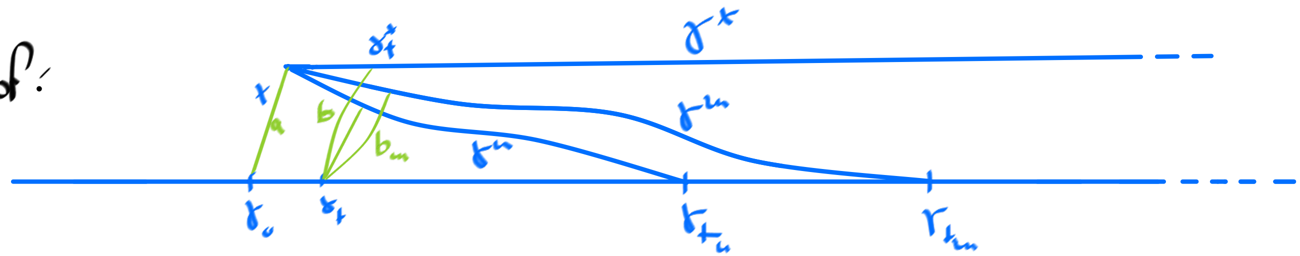
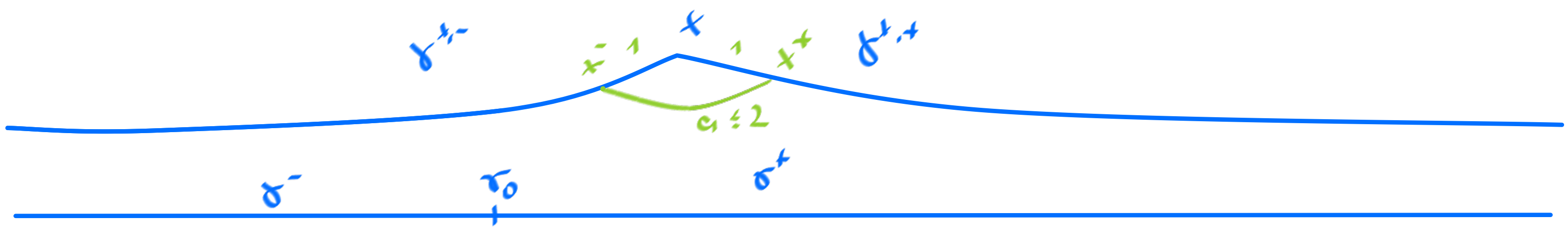


assume $\gamma^m \to \gamma^*$

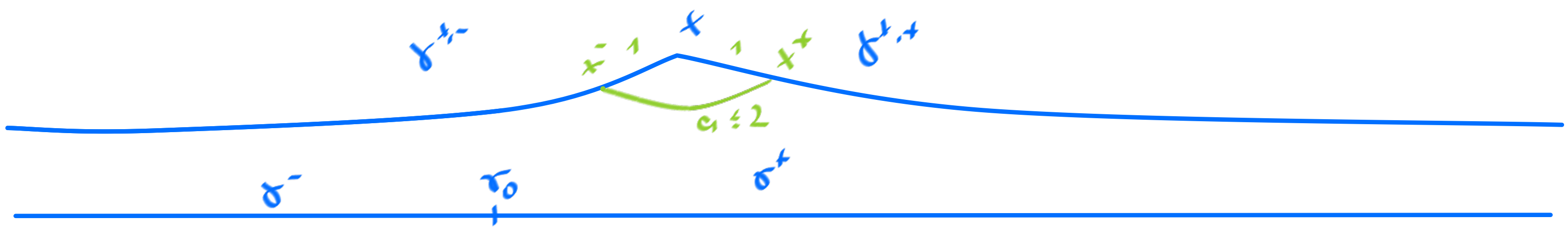
$\operatorname{curv}_{\text{Buse}} \geq 0$: $\quad b_m \geq \frac{t_m - t}{t_m} a \quad \xRightarrow{t_m \to \infty} \quad b \geq a$ (expansion)

Bi-Lipschitz Splitting Theorem (cont.)



- $b_{\gamma^\pm} : M \to \mathbb{R}$ is 1-Lipschitz, $b_{\gamma^\pm}(\gamma^{x,\pm}_t) = t + b_{\gamma^\pm}(x)$
- If $a = 2$ then $\gamma^{x,+} \cup \gamma^{x,-}$ is a line

Bi-Lipschitz Splitting Theorem (cont.)



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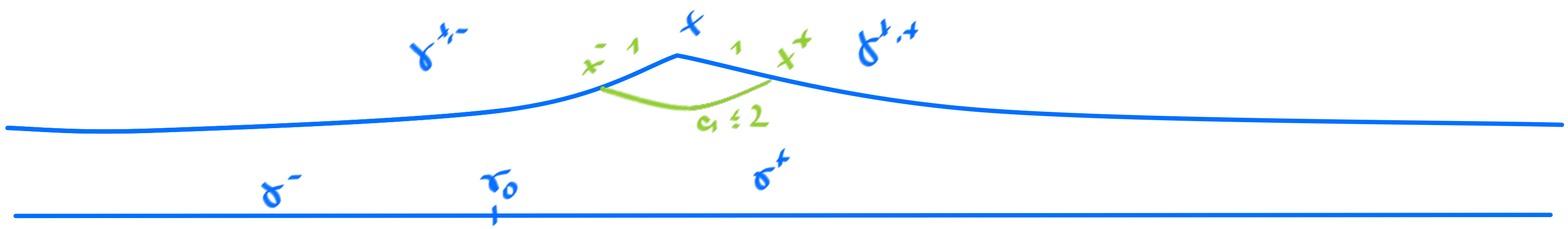
- suffices to show: $b_{\gamma^-}(x^+) \geq b_\gamma(x) - 1$

Bi-Lipschitz Splitting Theorem (cont.)



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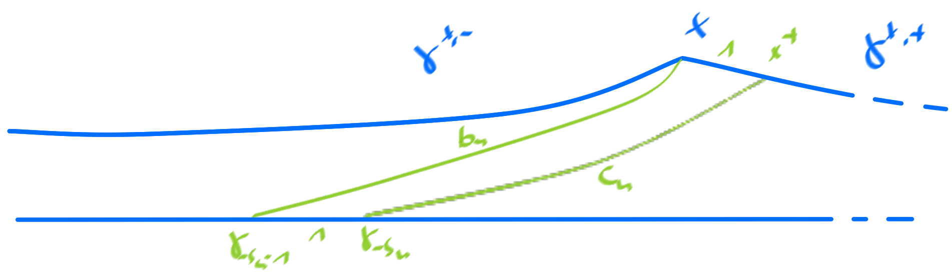
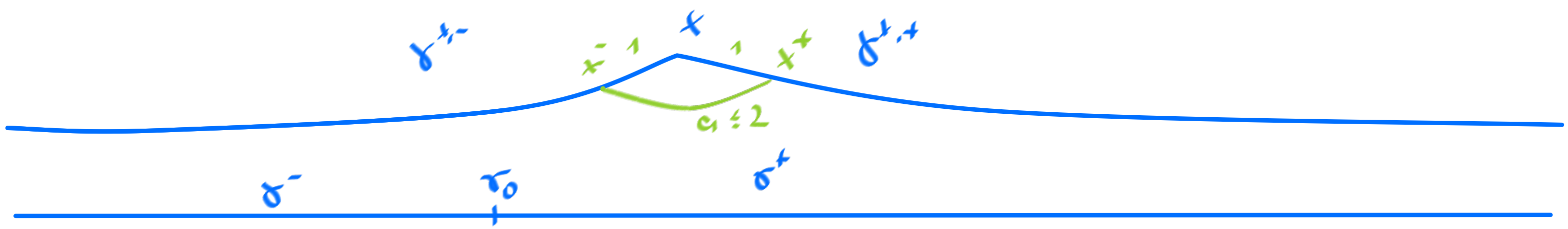


- from before $b_n \leq c_n$

Bi-Lipschitz Splitting Theorem (cont.)



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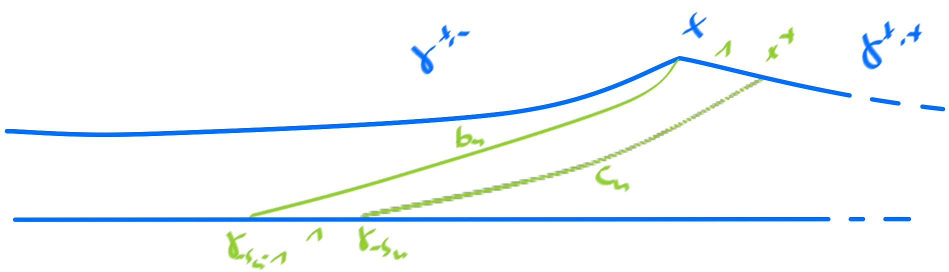
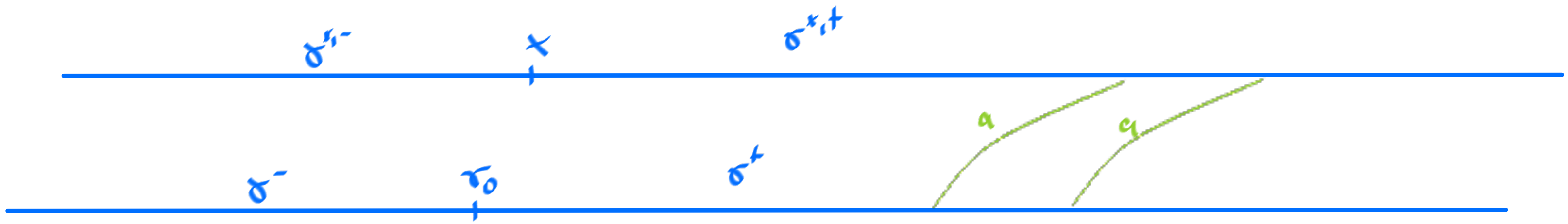


- from before $b_n \leq c_n$
- $n \to \infty$ $\quad b_{\gamma^-}(x) = \lim_{n\to\infty} s_n + 1 - b_n$
$$\geq \lim_{n\to\infty} s_n + 1 - c_n$$
$$= 1 + b_{\gamma^-}(x^+)$$

Bi-Lipschitz Splitting Theorem (cont.)



Isometric $\mathbb{R}$-action $\;\ldots\;$ $b_{\gamma^+}^{-1}(0)\times\mathbb{R}$, $d|_{b_{\gamma^+}^{-1}(0)} \otimes |\cdot - \cdot|$

is bi-Lipschitz to (M,d)

$\square$

Extensions:

$d(\gamma_t, \eta_t) \geq t^\lambda d(\gamma_0, \eta_0)$ $\lambda \geq 1$

$\rightsquigarrow$
- Splitting Theorem ✓
- non-trivial $\mathcal{H}^N \Rightarrow MCP(0, \lambda N)$ ✓
- no infinitesimal-to-global (yet) ✗

$\mathrm{curv}_{Buse} \geq k$

$\rightsquigarrow$
- $k > 0$:
  - Bonnet-Myers ✓
  - Max. Diam. Thm (top. suspension) ✓
- non-trivial $\mathcal{H}^N \Rightarrow MCP(k, N)$ ✓

Thank you
for your attention !