Quotient spaces & Ricci curvature

Martin Kell

(joint with F. Galaz-García, A. Mondino, G. Sosa)

$$\begin{tikzcd} G \arrow[r] & M \arrow[d] \\ & M^4 = M/G \end{tikzcd}$$

$$\begin{array}{ccc} G & \longrightarrow & M \\ & & \downarrow \\ & & M^* = M/G \end{array}$$

$$d_M(gx, gy) = d_M(x, y)$$

$$d^*(x^*, y^*) = d\big(G(x), G(y)\big) = \inf_{g \in G} d(gx, y)$$

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- $\sec_M \geq k \Rightarrow \sec_{M^*} \geq k$

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- (Pro - Wilhelm) $\exists$ warped product $M = B \times_f F$, F homogenous
  $\mathrm{Ric}_M > 0$, $\mathrm{Ric}_{M^*} = \mathrm{Ric}_B \not\geq 0$

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Facts true for metric foliations (eg. submetries)

$M^* = \{\mathcal{F}_x\}_{x \in M}$, $d^*(x^*, y^*) = d(\mathcal{F}_x, \mathcal{F}_y) \overset{!}{=} \inf_{y' \in \mathcal{F}_y} d(x, y')$

Warped Products

$(B, g_B), (F, g_F)$ Riemannian manifolds

$M = B \times_f F$ for $f : B \to \mathbb{R}$

$$g_M = g_B + e^{-\frac{2f}{n}} g_F$$

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Facts:

- $\{ \{x\} \times F \}_{x \in B}$ is a metric foliation
- if F is homogenous then $\mathrm{Isom}(F) \longrightarrow M$ with $(M^*, d^*) = (B, d_B)$
- $v, w \in TB \subset TM$ $N = \dim M$, $n = \dim B$

$$\mathrm{Ric}_M(v, w) := \mathrm{Ric}_B(v, w) + \mathrm{Hess}_{g_B} f(v, w) - \frac{df(v)\, df(w)}{N - n}$$

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Conclusion: If $\mathrm{Ric}_M \geq k$ then $\mathrm{Ric}^f_{B,N} \geq k$

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$$-\int \rho_t^{1-\frac{1}{N}} \, dm \leq -\int \tau_{K,N}^{1-t}(d(x,y))\, \rho_0^{-\frac{1}{N}}(x) + \tau_{K,N}^{t}(d(x,y))\, \rho_1^{-\frac{1}{N}}(y) \, d\pi(x,y)$$

where $\mu_t = \rho_t m + \mu_t^s := (e_t)_\# \sigma$, $\pi := (e_0, e_1)_\# \sigma$

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Assumption: G compact group with Haar measure μ_G

G acts isometrically, i.e. $d(gx, gy) = d(x,y)$

measure-preserving, i.e. $g_* m = m$

$$\mathcal{P}_2^G(M) := \{ \mu \in \mathcal{P}_2(M) : g_* \mu = \mu \ \forall g \in G \}$$

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Lifting measures & functions

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Lifting measures & functions

Lemma: $\hat{f}(x) := f(x^*)$; $f \in L^p(m^*) \Rightarrow \hat{f} \in L^p(m)$

$\forall \mu \in \mathcal{P}_2(M^*) \ \exists! \ \hat{\mu} \in \mathcal{P}_2^G(M) : p_*\hat{\mu} = \mu$

$$\mu = f m^* \Longrightarrow \hat{\mu} = \hat{f} m$$

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Theorem:

- $\Lambda : \mu \mapsto \hat{\mu}$ is an isometric embedding of $\mathcal{P}_2(M^*)$ into $\mathcal{P}_2(M)$
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Proposition: $\forall \mu, \nu \in \mathcal{P}_2(M^*)$, $\pi \in \mathrm{Opt}_2(\hat{\mu}, \hat{\nu})$

$$(x,y) \in \operatorname{supp}\pi \Rightarrow d(x,y) = d^*(x^*, y^*)$$

In particular, $(p,p)_*\pi \in \mathrm{Opt}(\mu,\nu)$.

and $\quad p_*\left(\mathrm{OptGeo}_2(\hat{\mu}, \hat{\nu})\right) = \mathrm{OptGeo}_2(\mu,\nu)$

Lemma: If $\varphi : M^* \to \mathbb{R} \cup \{-\infty\}$ is c_2-concave

then $\hat{\varphi} : M \to \mathbb{R} \cup \{-\infty\}$ is c_2-concave.

In particular, $(x,y) \in \partial^{c_2} \hat{\varphi} \Rightarrow d(x,y) = d^*(x^*, y^*)$.

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Pf: • $\hat{\varphi}(x) = \varphi(x^*)$

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- $\hat{\varphi}(x) + \hat{\psi}(y) = \varphi(x^*) + \psi(y^*)$

- $d(x,y) \geq d^*(x^*,y^*)$

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- $$\left.\begin{array}{l} \hat{\varphi}(x) + \hat{\psi}(y) = \varphi(x^*) + \psi(y^*) \\ d(x,y) \geq d^*(x^*,y^*) \end{array}\right\} (x,y) \in \partial \hat{\varphi} \Rightarrow \begin{array}{l} (x^*,y^*) \in \partial \varphi \\ d(x,y) = d^*(x^*,y^*) \end{array}$$

$\square$

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- $$\int d^2(x,y)\, d\hat{\pi}(x,y) = \int d^*(x^*,y^*)\, d\check{\pi}(x^*,y^*)$$
$$= \int \varphi\, d\mu + \int \psi\, d\nu = \int \hat{\varphi}\, d\hat{\mu} + \int \hat{\psi}\, d\hat{\nu}$$

□

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- $\tilde{\sigma} := \int g_* \sigma \, d\mu_G(g)$, $\tilde{\pi} := (e_0, e_1)_* \tilde{\sigma}$
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- $$\int \tau_{k,N}^{s}(d(x_0,x_1)) \, \hat{\rho}_i^{-\frac{1}{N}}(x_i) \, d\pi(x_0,x_1) = \int \tau_{k,N}^{s}(d^*(x_0^*,x_1^*)) \, \hat{\rho}_i^{-\frac{1}{N}}(x_i) \, d\pi(x_0,x_1)$$

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$$= \int \tau_{k,N}^{(s)}(d^*(x_0^*,x_1^*))\, \rho_i^{-\frac{1}{N}}(x_i^*)\, d\check\pi(x_0^*,x_1^*)$$

where $\check\pi := (p,p)_* \tilde\pi$. $\square$

Remark: "lifting is isometric embedding" only needs

- $d(F_x, F_y) = d(x, F_y)$
- $\exists \{\nu_{x^*}\}_{x^* \in M^*}$ with $\operatorname{supp} \nu_{x^*} \subset F_x$
  $W_2(\nu_{x^*}, \nu_{y^*}) = d^*(x^*, y^*)$

"Preservation of (distorted) Entropy" needs

$$m_{x^*} = \nu_{x^*} \quad \text{where} \quad m = \underbrace{\int m_{x^*} \, dm^*(x^*)}_{\text{disintegration over } p: M \to M^*}$$

$\rightsquigarrow$ metric measure foliation

Lots of new examples

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$(M_i, d_i, m_i)_{i=1,2}$ $RCD^*(N_i - 1, N_i)$

Join $M_1 * M_2 :=$ "Space of directions of $C_0(M_1) \times C_0(M_2)$"

$- RCD^*((N_1+N_2), N_1+N_2+1)$ [Gigli, Ketterer]

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Join $M_1 * M_2 :=$ "Space of directions of $C_0(M_1) \times C_0(M_2)$"

- $RCD^*((N_1+N_2), N_1+N_2+1)$ [Gigli, Kettlerer]
- not Alexandrov if $\sec_{M_i} \neq 1$
- If $\mathrm{Isom}(M_i)$ large then $\mathrm{Isom}(M_1 * M_2)$ "larger"

$\rightsquigarrow$ $M_1 * M_2 / G$ is CD^* $\forall\, G \leq \mathrm{Isom}(M_1 * M_2)$

(see later for RCD^*)

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Definition (Super Ricci flow)

(M, d_t, m_t) is a super N-Ricci flow if for a.e. $t\in(0,T)$ and all $\tau\mapsto\mu_\tau \ll m_t$ geodesic in $\mathcal{P}_2^{(t)}(M)$

$$\partial_\tau^+ \mathrm{Ent}_t(\mu_{1^-}) - \partial_\tau^- \mathrm{Ent}_t(\mu_{0^+})$$
$$\geq -\frac{1}{2}\partial_t^- W_t^2(\mu_0,\mu_1) + \frac{1}{N}\left|\mathrm{Ent}_t(\mu_1) - \mathrm{Ent}_t(\mu_0)\right|^2$$

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Theorem: The quotient of an equivariant super N-Ricci flow is a super N-Ricci flow.

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- (essentially) non-branching
- Good Transport Behavior (Existence of transport maps)
- MCP under
  - essentially non-branching [Cavalletti-Mondino]
  - $\forall x \neq y \in M : m(\{z \in M \mid d(x,z) = d(y,z)\}) = 0$ [K.]

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Question: How about the RCD?

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$$\operatorname{Lip} f = \operatorname{lip} f \quad m\text{-a.e.}$$

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Lemma: $\forall f \in \operatorname{Lip}(M^*), x \in M, r > 0: \operatorname{lip}^{M^*} f(x^*, r) = \operatorname{lip}^M \hat{f}(x,r)$.

Pf:

$$\sup_{y \in B_r(x)} \frac{|\hat{f}(y) - \hat{f}(x)|}{r} = \sup_{y \in B_r(x)} \frac{|f(y^*) - f(x^*)|}{r} = \sup_{y^* \in B_r(x^*)} \frac{|f(y^*) - f(x^*)|}{r}$$

$\square$

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Lipschitz function on quotient spaces

$$\mathrm{lip} f(x,r) := \sup_{y\in B_r(x)} \frac{|f(y)-f(x)|}{r}$$

$$\mathrm{lip}\, f(x) := \liminf_{r\to 0} \mathrm{lip} f(x,r)$$

$$\mathrm{Lip}\, f(x) := \limsup_{r\to 0} \mathrm{lip} f(x,r)$$

Def: (M,d,m) is a diff. space if for all Lipschitz functions f

$$\mathrm{Lip}\, f = \mathrm{lip}\, f \quad m\text{-a.e.}$$

"Lip-lip-condition"

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Corollary The lip-lip-condition is preserved under taking quotients.

Conclusion:

- If (M,d,m) is inf. Hilbertian and both (M,d,m) & (M^*,d^*,m^*) are PI-spaces then (M^*,d^*,m^*) is inf. Hilbertian.
- $RCD^*(K,N)$, $N\in[1,\infty)$, is preserved.

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More generally

Theorem: If (M,d,m) is geodesic then

$$i: W^{1,q}(M^*,m^*) \to W^{1,q}(M,m), \quad f \mapsto \hat{f}$$

is an isometric embedding onto the space of G-invariant Sobolev functions.

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More on RCD

If G acts Lipschitz & co-Lipschitz on orbits

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$$\dim_u(x) = \dim G(x) + \dim_u(x^*)$$

local dimension

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More on RCD

If G acts lipschitz & co-lipschitz on orbits

then

$$\dim n(x) = \dim G(x) + \dim n(x^*)$$

local dimension

$\rightsquigarrow$ Rigidity if $n(x^*) \in \{0,1,2\}$, i.e. $\dim G(x)$ large

Note: $\dim G = \dim G_x + \dim G(x)$

Principle Orbit Theorem

$$G_x = \{ g \in G \mid gx = x \}$$

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Theorem: If (M, d, m) has (GTB) then there is an (open) set $U \subset M$ of full measure such that

$\forall x, y \in U$: $G(x)$ & $G(y)$ are homeomorphic

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Pf: • $\varphi_x(y) := d^2(G(x), y) = (d^2(\bar{x}, \cdot))^{c}(y)$ is c_2-concave

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$\rightsquigarrow$ choose $x^* \in M$ with $\mathrm{type}(G_{x^*})$ minimal gives result $\square$

Cohomogeneity One Actions

Theorem: If (M, d, m) has (GTB) and $M^* \cong S^1$ then M is homeomorphic to an S^1-fiber bundle.

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- $\exists\, x_1, x_2, x_3, x_4 \in M.\ \forall y \in M: \ y \in U_{x_{i_y}} \cap U_{x_{j_y}}$ for $i_y, j_y \in \{1,2,3,4\}$

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$\rightsquigarrow$ $\cup U_{x_i} = M$, $G(y) \cong G(y')$

$\square$

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Orbifolds

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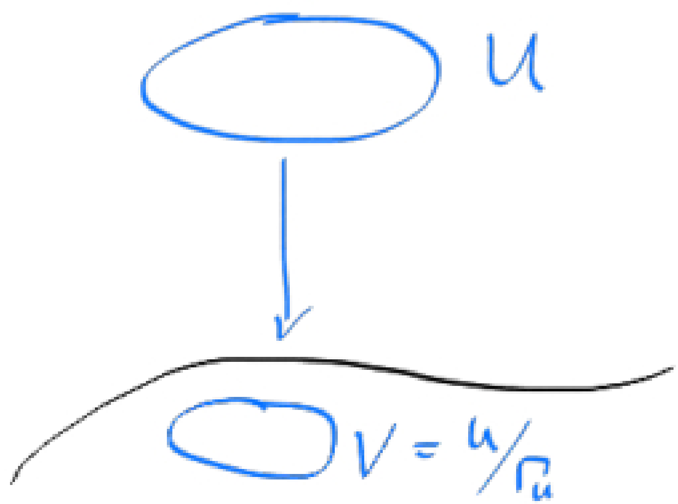
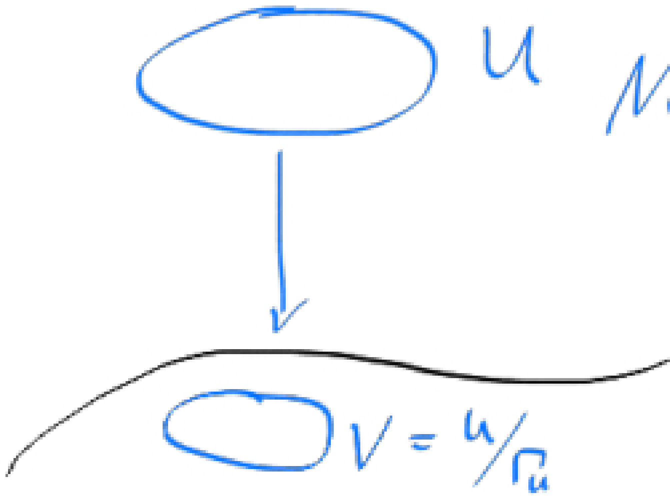
Orbifolds - spaces locally isometric to $\mathbb{R}^n / \Gamma_U$

$\leftarrow$ finite group depending on chart

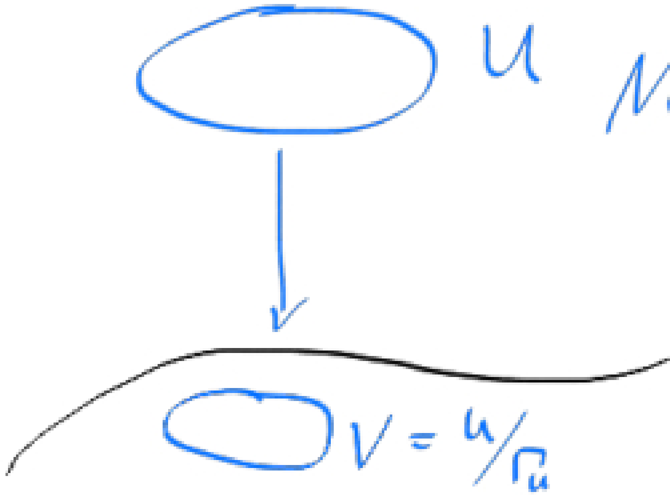


Orbifolds – spaces locally isometric to $\mathbb{R}^n/\Gamma_U$ ← finite group depending on chart



Note: · If U is locally $(R)CD^*$ then V is locally $(R)CD^*$

Orbifolds - spaces locally isometric to $\mathbb{R}^n/\Gamma_U$ ← finite group depending on chart



Note: • If U is locally $(R)CD^*$ then V is locally $(R)CD^*$

• Can choose V s.t. for (unique) $x_0 \in V$ $p^{-1}(x_0)$ is unique fixed point of Γ_U $(=\Gamma_{x_0})$ and Γ_U acts effectively (possibly $\Gamma_U = \{id\}$!)

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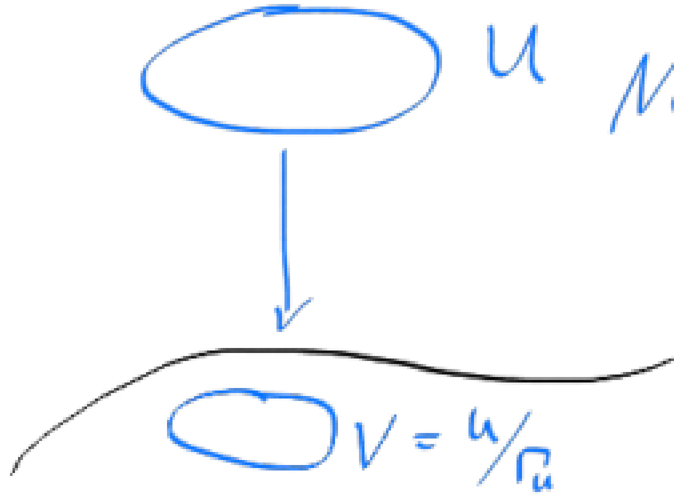

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Corollary (Bishop for orbifolds)

If $\mathrm{Ric}^{\mathcal{O}} \geq k$ then $\mathrm{vol}_{\mathcal{O}}(B_r(x)) \leq \frac{1}{|\Gamma_x|} V_{k,n}(r)$.

define either via $RCD^*(k,n)$ or $\mathrm{Ric} \geq k$ locally in U

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Application - Orbispaces & discrete infinite groups

Application – Orbispaces & discre infinite groups

Theorem: If (M, d, m) is locally isometric to a quotient of a (locally) RCD^*-space then it is an RCD^*-space.

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Application – Orbispaces & discre infinite groups

Theorem: If (M,d,m) is locally isometric to a quotient of a (locally) RCD*-space then it is an RCD*-space.

Definition: A discrete group acts *almost effectively* if

$$\forall x \in M: G_x = \{g \in G \mid gx = x\} \text{ is finite}$$

Theorem: If (M,d,m) has (GTB) and G acts effectively and alm. effectively then $G_x = \{id\}$ m-almost everywhere.

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Theorem: If (M,d,m) has (GTB) and G acts effectively and alm. effectively then $G_x = \{id\}$ m-almost everywhere.

Corollary: (M^*, d^*) is an orbispace which is loc. isometric to M/G_x. Furthermore, there is a unique m^* such that $m|_{U_x} = m^*|_{V_{x^*}}$ for all regular x, i.e. $G_x = \{id\}$.

Thank you!