1. Exercises "Toric geometry" SoSe 21

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Hand in on May 2 before 2pm via urm

Exercise 1: Let V be an affine variety and $f_1, \dots, f_s \in \mathbb{C}[V]$. They induce a polynomial map $\Phi : V \to \mathbb{C}^s$. The corresponding map on coordinate rings is given by

$$\Phi^* : \mathbb{C}[x_1, \dots, x_s] \to \mathbb{C}[V] : x_i \mapsto f_i.$$

Let $Y \subset \mathbb{C}^s$ be the Zariski closure of the image of Φ. Prove that $I(Y) = \mathrm{Ker}(\Phi^*)$.

Exercise 2: Let $\mathcal{A} = \{((4,0), (3,1), (1,3), (0,4)\} \subset \mathbb{R}^2$. Show that the ideal I of the affine toric variety $Y_{\mathcal{A}}$ defined by $\mathcal{A}$ (like in definition 1.10) is

$$I = \langle xw - yz, yw^2 - z^3, xz^2 - y^2 w, x^2 z - y^3 \rangle \subset \mathbb{C}[x, y, z, w].$$

You may use the fact that $V(I)$ is irreducible and of dimension 2 (this can be confirmed e.g. by using a Computeralgebrasystem like Singular).

Exercise 3: Let $\mathcal{A}_1 = \{(2,0), (1,1), (1,3)\} \subset \mathbb{R}^2$ and $\mathcal{A}_2 = \{(3,0), (1,1), (0,3)\} \subset \mathbb{R}^2$.

(1) Show that $Y := Y_{\mathcal{A}_1} = Y_{\mathcal{A}_2} = V(xz - y^3) \subset \mathbb{C}^3$.
(2) Regard $\Phi_{\mathcal{A}_i}$ as maps from $\mathbb{C}^2$ to Y. Show that $\Phi_{\mathcal{A}_2}$ is surjective and $\Phi_{\mathcal{A}_1}$ is not.

Exercise 4: Let T_N be a torus with character lattice M. Then every point $t \in T_N$ gives an evaluation map $\varphi_t : M \to \mathbb{C}^* : m \mapsto \chi^m(t)$. Show that φ_t is a group homomorphism and that the map $t \mapsto \varphi_t$ induces a group isomorphism $T_N \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$.