6. Exercises "Toric geometry" SoSe 21

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Hand in on June 13 before 2pm via urm

Exercise 1: Solve Exercise 1.3.12 from the book: show that the normalization of an affine toric variety in Prop. 1.3.8 is a toric morphism.

Exercise 2: Which of the following affine varieties is smooth, which is normal?

(1) $\mathrm{Spec}(\mathbb{C}[x^3y, x^2y, z])$
(2) $\mathrm{Spec}(\mathbb{C}[x, y, z, xyz^{-1}])$
(3) $\mathrm{Spec}(\mathbb{C}[x^5y^3z^2, x^{-1}yz^3])$

Exercise 3: Solve Exercise 3.9 from the book (prove Prop. 1.3.15): Suppose that we have strongly convex rational cones $\sigma_i \subset (N_i)_\mathbb{R}$ and a homomorphism $\overline{\phi} : N_1 \to N_2$. Then the induced map $\phi : T_{N_1} \to T_{N_2}$ extends to a map of affine toric varieties $\phi : U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\overline{\phi}_\mathbb{R}(\sigma_1) \subset \sigma_2$.

Exercise 4: Let S_1 be the semigroup generated by

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, -1)\} \subset \mathbb{Z}^3 =: M_1$$

and S_2 the semigroup generated by

$$\{(1, 0), (1, 1), (1, 2)\} \subset \mathbb{Z}^2 =: M_2.$$

Prove that the map from the dual lattice $N_1 = \mathrm{Hom}(M_1, \mathbb{Z}) = \mathbb{Z}^3$ to $N_2 = \mathrm{Hom}(M_2, \mathbb{Z}) = \mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

induces a morphism of toric varieties

$$\mathrm{Spec}(\mathbb{C}[S_1]) = V(xy - zw) \to V(ac - b^2) = \mathrm{Spec}(\mathbb{C}[S_2]).$$

Compute the induced map on toric varieties in the coordinates (x, y, z, w) and (a, b, c).