7. Exercises “Toric geometry” SoSe 21

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Exercise 1:

- Show that the homogeneous components h^k of the product $h = f \cdot g$ are given as $h^k = \sum_{i+j=k} f^i g^j$.
- Use this to show that an ideal $I = \langle f_1, \ldots, f_r \rangle$ given by homogeneous polynomials f_i is homogeneous (i.e. $f \in I$ implies the homogeneous components $f^i \in I$ for all i).

Exercise 2: Let $\#K = \infty$. Show that if f is not homogeneous, but f vanishes on all homogeneous coordinates of a point $x \in \mathbb{P}^n$, then each of the homogeneous components f^i vanishes at x. The following steps will help to prove this statement:

- Write f as a sum of its homogeneous components and evaluate f on $\lambda \cdot x$.
- Deduce that if f vanishes for all $\lambda \neq 0 \in K$, then $f^i(x) = 0$ for all i.

Exercise 3: Consider the curve $W = V(y^2 - z, y^3 - w) \subset \mathbb{R}^3$.

- If we parametrize W by (t, t^2, t^3), show that as $t \to \pm\infty$, the point $(1 : t : t^2 : t^3) \in \mathbb{P}^3_{\mathbb{R}}$ approaches $(0 : 0 : 0 : 1)$. Thus we expect W to have one point at infinity.
- Let $V' = V(xz - y^2, x^2w - y^3, yw - z^2) \subset \mathbb{P}^3_{\mathbb{R}}$. Show that $V' \cap U_0 = W$ and $V' \cap V(x) = \{(0 : 0 : 0 : 1)\}$.

Exercise 4: The Segre-map $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ sends a point (x, y) to all products of components of x and components of y:

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m} : ((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) \mapsto ((x_i y_j)_{ij}).$$

- Show that σ is a well-defined and injective map.
- For $n = m = 1$, show that the image of the Segre-map is a projective variety.

Notice that the Segre-map is used in general to show that products of projective varieties are projective varieties again.