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Fachbereich Mathematik
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Algebraic Number Theory

SS 2026

Sheet 1

If you want your solutions to be corrected, upload them on URM by the end of April 27.
Write your name and matriculation number on each sheet. If you upload the exercises as a group, only one person can do it instead of the whole group.
The exercises marked with $*$ might be more complicated or involved than the others.

Exercise 1.1 Show that an element $\alpha \in \mathbf{Z}[i]$ is irreducible in $\mathbf{Z}[i]$ if and only if $N(\alpha)$ is prime in $\mathbf{Z}$ or if $\alpha = u \cdot p$ where $u \in \mathbf{Z}[i]^\times$ and $p \in \mathbf{Z}$ is a prime number such that $p \equiv 3 \pmod 4$.

Exercise 1.2 Let R be an UFD and let $a, b \in R$.

(i) Assume that $a, b \in R$ are coprime and that that there is $c \in R, n \in \mathbf{Z}_{>0}$ such that $ab = c^n$. Show that there are $u \in R^\times, a', b' \in R$ such that $a = u \cdot (a')^n, b = u^{-1} \cdot (b')^n$.

(ii) Is the previous statement true if a, b are not necessarily coprime? Prove it or give a counterexample.

(iii) * Using the ring $R = \mathbf{Z}[i]$, find all $x, y \in \mathbf{Z}$ such that $y^2 = x^3 - 1$.

Exercise 1.3 We have seen in the lectures that the ring $\mathbf{Z}[i]$ is an UFD, here we want to describe a similar ring, that is not an UFD.

(i) Prove that $\mathbf{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbf{Z}\}$ is a subring of $\mathbf{C}$.

(ii) Consider the norm map $N \colon \mathbf{Z}[\sqrt{-5}] \to \mathbf{Z}$ defined by $N(a + b\sqrt{-5}) = a^2 + 5b^2$. Prove that $N(\alpha\beta) = N(\alpha)N(\beta)$ for every $\alpha, \beta \in \mathbf{Z}[\sqrt{-5}]$. Write down all the invertible elements of $\mathbf{Z}[\sqrt{-5}]$.

(iii) Show that the element 6 has two distinct factorizations into irreducibles in $\mathbf{Z}[\sqrt{-5}]$.