

Sheet 10

If you want your solutions to be corrected, upload them on URM by the end of July 6.
 Write your name and matriculation number on each sheet. If you upload the exercises as a group,
 only one person can do it instead of the whole group.

The exercises marked with * might be more complicated or involved than the others.

Exercise 10.1 Let K, L be two number fields and denote by KL the smallest field containing both of them. For example, if $K = \mathbf{Q}(\alpha), L = \mathbf{Q}(\beta)$, then $KL = \mathbf{Q}(\alpha, \beta)$. Consider also the set

$$\mathcal{O}_K \mathcal{O}_L = \left\{ \sum_{i=1}^n \alpha_i \beta_i \mid \alpha_i \in \mathcal{O}_K, \beta_i \in \mathcal{O}_L, n \in \mathbf{N} \right\} \subseteq \mathcal{O}_{KL}$$

This is a subring of \mathcal{O}_{KL} : for example if $\mathcal{O}_K = \mathbf{Z}[\alpha], \mathcal{O}_L = \mathbf{Z}[\beta]$, then $\mathcal{O}_K \mathcal{O}_L = \mathbf{Z}[\alpha, \beta]$.

Assume that $[KL : \mathbf{Q}] = [K : \mathbf{Q}][L : \mathbf{Q}]$ and let $d = \text{GCD}(\text{disc } K, \text{disc } L)$. We want to show that $d \cdot \mathcal{O}_{KL} \subseteq \mathcal{O}_K \mathcal{O}_L$.

- (i) Show that if $\sigma_1, \dots, \sigma_n : K \hookrightarrow \mathbf{C}$ are all the embeddings of K and $\tau_1, \dots, \tau_m : L \hookrightarrow \mathbf{C}$ are all the embeddings of L , then for any $i = 1, \dots, n$ and $j = 1, \dots, m$ there is a unique embedding $\eta_{ij} : KL \hookrightarrow \mathbf{C}$ such that $\eta_{ij}|_K = \sigma_i, \eta_{ij}|_L = \tau_j$. Show also that the η_{ij} are all the embeddings of KL .
- (ii) Assume that $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_m)$ are integral bases of $\mathcal{O}_K, \mathcal{O}_L$ respectively. Fix an element $\gamma \in \mathcal{O}_{KL}$ and show that can be written as

$$\gamma = \sum_{i,j} a_{ij} \alpha_i \beta_j \quad \text{for certain } a_{hk} \in \mathbf{Q}$$

- (iii) Consider the $n \times n$ matrix $S = (\sigma_i(\alpha_h))_{1 \leq i, h \leq n}$ and define $x_i = \sum_j a_{ij} \beta_j$, for all $i = 1, \dots, n$. Show that $\det(S) \cdot x_i$ is integral over \mathbf{Z} and then deduce that $\text{disc}(K) \cdot x_i \in \mathcal{O}_L$ for all i (Hint: apply appropriate η_{hk} to the previous equality for γ).
- (iv) Show that $\text{disc}(K) a_{ij}, \text{disc}(L) a_{ij} \in \mathbf{Z}$ for all i, j . Conclude that $d \cdot \mathcal{O}_{KL} \subseteq \mathcal{O}_K \mathcal{O}_L$.

Exercise 10.2 We consider the rings of integers of cyclotomic fields.

- (i) Assume that $n = p^k$ for a certain prime $p > 0$ and $k \in \mathbf{Z}$. Show that $\mathcal{O}_{\mathbf{Q}(\zeta_{p^k})} = \mathbf{Z}[\zeta_{p^k}]$ (Hint: the same proof of the case $n = p$ works here with some adjustments).
- (ii) Assume that n, m are coprime integers with $\mathcal{O}_{\mathbf{Q}(\zeta_n)} = \mathbf{Z}[\zeta_n], \mathcal{O}_{\mathbf{Q}(\zeta_m)} = \mathbf{Z}[\zeta_m]$. Show that $\mathcal{O}_{\mathbf{Q}(\zeta_{nm})} = \mathbf{Z}[\zeta_{nm}]$.
- (iii) Conclude that $\mathcal{O}_{\mathbf{Q}(\zeta_n)} = \mathbf{Z}[\zeta_n]$ for all n .