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Fachbereich Mathematik
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Algebraic Number Theory
SS 2026

Sheet 3

If you want your solutions to be corrected, upload them on URM by the end of May 11.
Write your name and matriculation number on each sheet. If you upload the exercises as a group, only one person can do it instead of the whole group.
The exercises marked with $*$ might be more complicated or involved than the others.

For the next exercise you can use a fact mentioned in the second exercise sheet: if $d, e \in \mathbf{Z}$ are two square-free integers, then $\mathbf{Q}(\sqrt{d}) = \mathbf{Q}(\sqrt{e})$ if and only if $d = e$.

Exercise 3.1 Let $d, e \in \mathbf{Z}$ be two distinct square-free integers and consider the number field $K = \mathbf{Q}(\sqrt{d}, \sqrt{e})$.

(i) Compute a basis of K over $\mathbf{Q}(\sqrt{d})$ and a basis of K over $\mathbf{Q}$.

(ii) Show that any element $\alpha \in K$ can be written as $\alpha = \beta + \gamma \cdot \sqrt{e}$ for some unique $\beta, \gamma \in \mathbf{Q}(\sqrt{d})$. Show also that

$$\mathrm{Tr}_{K/\mathbf{Q}(\sqrt{d})}(\alpha) = 2\beta, \quad \mathrm{N}_{K/\mathbf{Q}(\sqrt{d})}(\alpha) = \beta^2 - e\gamma^2$$

(iii) Let $\mathbf{Q} \subseteq F \subset K$ be a subfield with $[F : \mathbf{Q}] = 2$ and let $\alpha \in K$. Show that $\alpha \in \mathcal{O}_K$ if and only if $\mathrm{N}_{K/F}(\alpha) \in \mathcal{O}_F$ and $\mathrm{Tr}_{K/F}(\alpha) \in \mathcal{O}_F$.

Consider now the number field $K := \mathbf{Q}(\sqrt{2}, \sqrt{3})$.

(iv) Show that every $\alpha \in \mathcal{O}_K$ can be written in the form

$$\alpha = a + b\sqrt{3} + c\sqrt{2} + d\frac{\sqrt{2} + \sqrt{6}}{2}$$

where $a, b, c, d \in \mathbf{Z}$. Conclude that an integral basis of $\mathcal{O}_K$ is given by

$$1, \sqrt{3}, \sqrt{2}, \frac{\sqrt{2} + \sqrt{6}}{2}.$$

(Hint: use point (iii) with respect to three distinct subfields $F \subseteq K$).