Universität Tübingen
Fachbereich Mathematik
D. Agostini, K. Heesterbeek, C. Nollau

Algebraic Number Theory

SS 2026

Sheet 4

If you want your solutions to be corrected, upload them on URM by the end of May 18.
Write your name and matriculation number on each sheet. If you upload the exercises as a group, only one person can do it instead of the whole group.
The exercises marked with $*$ might be more complicated or involved than the others.

Exercise 4.1 Let K be a number field and $\mathcal{O}_K$ its ring of integers.

(i) For any $\alpha_1, \dots, \alpha_m \in K$ consider the matrix $T(\alpha_1, \dots, \alpha_m) = (\mathrm{Tr}_{K/\mathbf{Q}}(\alpha_i\alpha_j))_{1\le i,j\le m}$. Prove that if $\det T(\alpha_1, \dots, \alpha_m) \neq 0$ then the $\alpha_1, \dots, \alpha_m$ are linearly independent over $\mathbf{Q}$. Is the converse true? Prove it if it is true or give a counterexample if it is false.

(ii) Let $R \subseteq \mathcal{O}_K$ be a subring. Is it true that R is a lattice in $\mathcal{O}_K$, meaning that it is a free abelian group of rank $[K : \mathbf{Q}]$? Prove it if it is true or give a counterexample if it is false.

(iii) Let $\alpha \in \mathcal{O}_K$ be a non-zero element and consider the principal ideal $(\alpha) \subseteq \mathcal{O}_K$. We know that the quotient $\mathcal{O}_K/(\alpha)$ is finite. Prove that

$$|\mathcal{O}_K/(\alpha)| = |\,\mathrm{N}_{K/\mathbf{Q}}(\alpha)|$$

Prove then that if $\mathrm{N}_{K/\mathbf{Q}}(\alpha)$ is a prime number, then (α) is a prime ideal. Is the converse also true? Prove it if it is true and give a counterexample otherwise.

Exercise 4.2 Let $K = \mathbf{Q}(\alpha)$ be a number field of degree n. Let $m_\alpha(x)$ be the minimal polynomial of α and let $\sigma_1, \dots, \sigma_n : K \hookrightarrow \mathbf{C}$ be all the embeddings of K. Prove that

$$\mathrm{disc}(1, \alpha, \dots, \alpha^{n-1}) = \prod_{i<j} (\sigma_i(\alpha) - \sigma_j(\alpha))^2 = (-1)^{\frac{n(n-1)}{2}} N_{K/\mathbf{Q}}(m'_\alpha(\alpha)).$$

Exercise 4.3

(i) Let $a, b \in \mathbf{Z}$. Assume that the polynomial $f(x) = x^3 + ax + b$ is irreducible over $\mathbf{Q}$, let α be a root of $f(x)$ and $K = \mathbf{Q}(\alpha)$. Show that $\mathrm{disc}(1, \alpha, \alpha^2) = -(4a^3 + 27b^2)$.

(ii) Let $g(x) = x^3 + x + 1$. Prove that this polynomial is irreducible over $\mathbf{Q}$.

(iii) If α is a root of $g(x)$ and $K = \mathbf{Q}(\alpha)$, find an integral basis of $\mathcal{O}_K$.