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Fachbereich Mathematik
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Algebraic Number Theory

SS 2026

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Sheet 6

This exercise sheet will be discussed in class.

The following fact will be proved later on in the lectures:

Fact: *If a number field K has discriminant* $\operatorname{disc}(K) = \pm 1$, *then* $K = \mathbf{Q}$.

Use this fact to solve the second point of the following exercise:

Exercise 6.1

(i) Let $F \subseteq K$ be two number fields, in particular $\mathcal{O}_F \subseteq \mathcal{O}_K$. For any ideal $I \subseteq \mathcal{O}_F$ denote by $I\mathcal{O}_K \subseteq \mathcal{O}_K$ the ideal that it generates inside $\mathcal{O}_K$. If $I = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_r^{e_r}$ is the prime factorization of I in $\mathcal{O}_K$, show that $I\mathcal{O}_K = (\mathfrak{p}_1\mathcal{O}_K)^{e_1} \dots (\mathfrak{p}_r\mathcal{O}_K)^{e_r}$ inside $\mathcal{O}_K$.

(ii) Let $K, L \subseteq \mathbf{C}$ be two number fields such that $\operatorname{disc}(K)$ and $\operatorname{disc}(L)$ are coprime. Show that $K \cap L = \mathbf{Q}$.

Exercise 6.2 Consider the number field $K = \mathbf{Q}(\sqrt{-5})$ and its ring of integers $\mathcal{O}_K$. Compute the prime factorization of the ideals (120) and $(120, 2\sqrt{-5} - 7)$ in $\mathcal{O}_K$.

Exercise 6.3 Let A be a domain with fraction field $F = \operatorname{Frac} A$. Fix a prime ideal $\mathfrak{p} \subseteq A$ and define the subset of F given by

$$A_{\mathfrak{p}} = \left\{ \frac{a}{s} \mid a \in A, s \in A \setminus \mathfrak{p} \right\}$$

This is called the *localization of A at $\mathfrak{p}$*. If $I \subseteq A$ is an ideal we also define

$$I_{\mathfrak{p}} = \left\{ \frac{a}{s} \mid a \in I, s \in A \setminus \mathfrak{p} \right\} \subseteq A_{\mathfrak{p}}.$$

(i) Show that $A \subseteq A_{\mathfrak{p}}$ is a subring of F and that $I_{\mathfrak{p}}$ is an ideal in $A_{\mathfrak{p}}$.

(ii) Prove that $I_{\mathfrak{p}}$ is a proper ideal of $A_{\mathfrak{p}}$ if and only if $I \subseteq \mathfrak{p}$. In this case and if I is prime show that $I = I_{\mathfrak{p}} \cap A$. Conclude that there is a bijection:

$$\{\text{prime ideals of } A \text{ contained in } \mathfrak{p}\} \to \{\text{prime ideals in } A_{\mathfrak{p}}\}, \quad I \mapsto I_{\mathfrak{p}}.$$

Show also that $\mathfrak{p}_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$.

(iii) Show that if A is Noetherian then $A_{\mathfrak{p}}$ is Noetherian as well.

(iv) Show that $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ where the intersection runs along all maximal ideals in A.

(v) Assume that A is Noetherian and prove that A is a Dedekind domain if and only if $A_{\mathfrak{m}}$ is a discrete valuation ring for each maximal ideal $\mathfrak{m} \subseteq A$.
