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Fachbereich Mathematik
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Algebraic Number Theory
SS 2026

Sheet 9

If you want your solutions to be corrected, upload them on URM by the end of June 29.
Write your name and matriculation number on each sheet. If you upload the exercises as a group, only one person can do it instead of the whole group.
The exercises marked with $*$ might be more complicated or involved than the others.

Exercise 9.1 Fix a square-free positive integer $d \in \mathbf{Z}, d > 0$ and consider the corresponding *Pell equation*:

$$x^2 - dy^2 = 1.$$

A trivial solution to this equation is given by $(x, y) = (\pm 1, 0)$. An integer positive solution is a solution where both x, y are integer and positive. Notice that any integer non-trivial solution can be made into a positive one by changing the signs of (x, y).

(i) Show that the Pell equation has infinitely many integral solutions (x, y) (You can restrict to the case $d \equiv 2, 3 \pmod 4$. The case $d \equiv 1 \pmod 4$ is a bit harder, but you are anyway encouraged to think about it).

(ii) If $d = 3$, find three distinct positive integral solutions to the Pell equation.

Consider now the modified Pell equation

$$x^2 - dy^2 = -1.$$

(iii) Show that if the modified Pell equation has one integer solution, then it has infinitely many.

(iv) Give a concrete example of a modified Pell equation with infinitely many integer solutions and another one with no integer solutions.

Exercise 9.2 Consider a number field K and assume that it is a normal extension of $\mathbf{Q}$ of degree $n = [K : \mathbf{Q}]$. Let also $\alpha \in K$ be such that $K = \mathbf{Q}(\alpha)$ and consider $d = \operatorname{disc}(1, \alpha, \ldots, \alpha^{n-1})$.

(i) Show that $\sqrt{d} \in K$.

Let now $p \in \mathbf{Z}$ be an odd prime number and consider the cyclotomic field $K = \mathbf{Q}(\zeta_p)$.

(ii) Show that K is a Galois extension of $\mathbf{Q}$ with Galois group isomorphic to $\mathbf{Z}/(p-1)\mathbf{Z}$.

(iii) Show that there is a unique quadratic extension L of $\mathbf{Q}$ contained in K and find an explicit $e \in \mathbf{Z}$ such that $L = \mathbf{Q}(\sqrt{e})$.