**1. Exercises "Toric geometry" SoSe 26**

Parisa Ebrahimian, Veronika Körber, Hannah Markwig

**Hand in on April 21 before 12pm**

**Exercise 1:** Let $V$ be an affine variety and $f_1, \ldots, f_s \in \mathbb{C}[V]$. They induce a polynomial map $\Phi : V \to \mathbb{C}^s$. The corresponding map on coordinate rings is given by

$$\Phi^* : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}[V] : x_i \mapsto f_i.$$

Let $Y \subset \mathbb{C}^s$ be the Zariski closure of the image of $\Phi$. Prove that $I(Y) = \mathrm{Ker}(\Phi^*)$.

**Exercise 2:** Let $\mathcal{A} = \{((4,0),(3,1),(1,3),(0,4)\} \subset \mathbb{R}^2$. Show that the ideal $I$ of the affine toric variety $Y_{\mathcal{A}}$ defined by $\mathcal{A}$ (like in definition 1.10) is

$$I = \langle xw - yz, yw^2 - z^3, xz^2 - y^2w, x^2z - y^3 \rangle \subset \mathbb{C}[x,y,z,w].$$

You may use the fact that $V(I)$ is irreducible and of dimension 2 (this can be confirmed e.g. by using a Computeralgebrasystem like SINGULAR).

**Exercise 3:** Let $\mathcal{A}_1 = \{(2,0),(1,1),(1,3)\} \subset \mathbb{R}^2$ and $\mathcal{A}_2 = \{(3,0),(1,1),(0,3)\} \subset \mathbb{R}^2$.

(1) Show that $Y := Y_{\mathcal{A}_1} = Y_{\mathcal{A}_2} = V(xz - y^3) \subset \mathbb{C}^3$.
(2) Regard $\Phi_{\mathcal{A}_i}$ as maps from $\mathbb{C}^2$ to $Y$. Show that $\Phi_{\mathcal{A}_2}$ is surjective and $\Phi_{\mathcal{A}_1}$ is not.

**Exercise 4:** Let $T_N$ be a torus with character lattice $M$. Then every point $t \in T_N$ gives an evaluation map $\varphi_t : M \to \mathbb{C}^* : m \mapsto \chi^m(t)$. Show that $\varphi_t$ is a group homomorphism and that the map $t \mapsto \varphi_t$ induces a group isomorphism $T_N \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$.