6. Exercises "Toric geometry" SoSe 26

Parisa Ebrahimian, Veronika Körber, Hannah Markwig

Hand in on June 2 before 12pm

Exercise 1: Which of the following affine varieties is smooth, which is normal?

(1) $\mathrm{Spec}(\mathbb{C}[x^3y, x^2y, z])$
(2) $\mathrm{Spec}(\mathbb{C}[x, y, z, xyz^{-1}])$
(3) $\mathrm{Spec}(\mathbb{C}[x^5y^3z^2, x^{-1}yz^3])$

Exercise 2:
Let S_1 be the semigroup generated by

$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,-1)\} \subset \mathbb{Z}^3 =: M_1$$

and S_2 the semigroup generated by

$$\{(1,0),(1,1),(1,2)\} \subset \mathbb{Z}^2 =: M_2.$$

Prove that the map from the dual lattice $N_1 = \mathrm{Hom}(M_1, \mathbb{Z}) = \mathbb{Z}^3$ to $N_2 = \mathrm{Hom}(M_2, \mathbb{Z}) = \mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

induces a morphism of toric varieties

$$m - \mathrm{Spec}(\mathbb{C}[S_1]) = V(xy - zw) \to V(ac - b^2) = m - \mathrm{Spec}(\mathbb{C}[S_2]).$$

Compute the induced map on toric varieties in the coordinates (x, y, z, w) and (a, b, c).

Exercise 3:

- Show that the homogeneous components h^k of the product $h = f \cdot g$ are given as $h^k = \sum_{i+j=k} f^i g^j$.
- Use this to show that an ideal $I = \langle f_1, \ldots, f_r \rangle$ given by homogeneous polynomials f_i is homogeneous (i.e. $f \in I$ implies the homogeneous components $f^i \in I$ for all i).

Exercise 4: Let $\#K = \infty$. Show that if f is not homogeneous, but f vanishes on all homogeneous coordinates of a point $x \in \mathbb{P}^n$, then each of the homogeneous components f^i vanishes at x. The following steps will help to prove this statement:

- Write f as a sum of its homogeneous components and evaluate f on $\lambda \cdot x$.
- Deduce that if f vanishes for all $\lambda \neq 0 \in K$, then $f^i(x) = 0$ for all i.