7. Exercises "Toric geometry" SoSe 26

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Exercise 1: Consider the curve $W = V(y^2 - z, y^3 - w) \subset \mathbb{R}^3$.

- If we parametrize W by (t, t^2, t^3), show that as $t \to \pm\infty$, the point $(1 : t : t^2 : t^3) \in \mathbb{P}^3_{\mathbb{R}}$ approaches $(0 : 0 : 0 : 1)$. Thus we expect W to have one point at infinity.
- Let $V' = V(xz - y^2, x^2w - y^3, yw - z^2) \subset \mathbb{P}^3_{\mathbb{R}}$. Show that $V' \cap U_0 = W$ and $V' \cap V(x) = \{(0 : 0 : 0 : 1)\}$.

Exercise 2:(See exercise 2.0.5) The Segre-map $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ sends a point (x, y) to all products of components of x and components of y:

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m} : ((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) \mapsto ((x_i y_j)_{ij}).$$

- Show that σ is a well-defined and injective map.
- For $n = m = 1$, show that the image of the Segre-map is a projective variety.

Notice that the Segre-map is used in general to show that products of projective varieties are projective varieties again.

Exercise 3: (See exercise 2.1.2) Let $\mathcal{A} = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n$ consist of the columns of an $n \times s$ matrix A with integer entries. Show: there exists a $u \in \mathbb{Z}^n$ and $k \in \mathbb{N} \setminus \{0\}$ with $\langle m_i, u \rangle = k$ for all i if and only if $(1, \ldots, 1)$ is in the rowspace of A.

Exercise 4: (See exercise 2.1.3) Given a finite set $\mathcal{A} \subset M$, prove that the rank of $\mathbb{Z}'\mathcal{A}$ equals the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathcal{A}$.