Eberhard-Karls-Universität Tübingen
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Wintersemester 2020/2021
4. November 2020

Introduction to Commutative Algebra and Algebraic Geometry
Solution - Exercise Sheet 1

Exercise 1.
Let R be a commutative ring with one. For a subset $S \subset R$ the ideal generated by S is defined as the smallest ideal of R containing S:

$$\langle S \rangle := \bigcap_{\substack{J \text{ ideal of } R, \\ S \subset J}} J.$$

Show:

$$\langle S \rangle = \{\sum_{i=1}^{n} \lambda_i s_i \mid n \in \mathbb{N},\, \lambda_i \in R,\, s_i \in S \;\forall i \in \{1, ..., n\}\}.$$

Proof: Let $K := \{\sum_{i=1}^{n} \lambda_i s_i \mid n \in \mathbb{N},\, \lambda_i \in R,\, s_i \in S \;\forall i \in \{1, ..., n\}\}$. Then K is an ideal in R, since $K \neq \emptyset$, $K \subset R$ and for $x, y \in K, \lambda \in R$: $\exists n, m \in \mathbb{N}, \lambda_i, \mu_j \in R, s_i, t_j \in S \;\forall i = 1..., n, j = 1, ..., m$: $x = \sum_{i=1}^{n} \lambda_i s_i$, $y = \sum_{j=1}^{m} \mu_j t_j$ and thus: $x + \lambda y = \sum_{i=1}^{n} \lambda_i u_i + \sum_{j=1}^{m} \lambda \mu_j v_j$.
Therefore: $x + \lambda y \in K$. So K is an ideal in R.

Since the intersection of ideals is an ideal (the group properties are contained as well as the closure under multiplication with elements of R), i.e. $\langle S \rangle_R$ is also an ideal of R.

„$\subset$": $S \subset K = \{\sum_{i=1}^{n} \lambda_i s_i \mid n \in \mathbb{N},\, \lambda_i \in R,\, s_i \in S \;\forall i \in \{1, ..., n\}\}$ through $n = 1, \lambda_1 = 1$. Since K is an ideal, which contains S: we have $\bigcap_{J \leq R, S \subset J} J \subset K$
$\Rightarrow \langle S \rangle_R \subset K$.

„$\supset$": We know that for all $x, y \in \langle S \rangle_R$ and $\lambda \in R$ we have $x + \lambda y \in \langle S \rangle_R$, because $\langle S \rangle_R$ is an ideal in R. In particular we have $S \subset \langle S \rangle_R$.
$\Rightarrow \forall n \in \mathbb{N}, \lambda_i \in R, s_i \in S \,\forall i = 1, ..., n : \sum_{i=1}^{n} \lambda_i s_i \in \langle S \rangle_R$.
$\Rightarrow K \subset \langle S \rangle_R$ ☐

Exercise 2.
Let K be a field and let $R = K[x]$. Let $I = aR$, $J = bR \subset K[x]$ be two principal ideals. Show

$$I + J = \langle \text{gcd}(a, b) \rangle$$

$$I \cap J = \langle \text{lcm}(a, b) \rangle,$$

where gcd denotes the greatest common divisor and lcm the least common multiple.

Proof: Let $c = \text{lcm}(a, b)$ and $d = \text{gcd}(a, b)$.
We know that $d|a$ and $d|b$, so there exist $u, v \in R$ such that $a = ud$ and $b = vd$ thus $I = aR$ and $J = bR$ are contained in $dR = \langle \text{gcd}(a, b) \rangle$. It follows that $I + J \subset \langle \text{gcd}(a, b) \rangle$.
Since $R = K[x]$ is an euclidean ring, there exist $n, m \in R$ such that $d = na + mb$. It follows that $d \in I + J$ and thus $I + J = \langle \text{gcd}(a, b) \rangle$.

We know that $a|c$ and $b|c$, so there exist $u, v \in R$ such that $c = ua$ and $c = vb$ thus $I = aR$ and $J = bR$ both contain $cR = \langle \text{lcm}(a, b) \rangle$. It follows that $I \cap J \supset \langle \text{lcm}(a, b) \rangle$.
Let $s \in I \cap J$. Then, there exists $r_1, r_2 \in R$ such that $s = ar_1 = br_2$. Since c is the least common multiple of a and b it follows that $c|s$ and thus there exists $t \in R$ such that $s = ct \in \langle \text{lcm}(a, b) \rangle$ Hence $I \cap J = \langle \text{lcm}(a, b) \rangle$. ☐

Exercise 3.
Let R be a commutative ring with one. Let $I \subset R$ be an ideal. Show that $\sqrt{I}$ is an ideal in R.

Proof: We have $\sqrt{I} = \{r \in R | \exists n \in \mathbb{N} \text{such that} r^n \in I\}$. Since $0 \in I$, we know $0 \in \sqrt{I}$ so $\sqrt{I} \neq \emptyset$.

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Let $a, b \in \sqrt{I}$ and let $\lambda \in R$. There exist $n, m \in \mathbb{N}$, such that $a^n, b^m \in I$.
Then $(\lambda a)^n = \lambda^n a^n \in I$ and thus $\lambda a \in \sqrt{I}$. It also follows that

$$(a+b)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} a^i b^{n+m-i} \in I,$$

because for each $i \in \{0, ..., n+m\}$ either a^i or b^{n+m-i} is contained in I. And thus $a+b \in \sqrt{I}$. Hence $\sqrt{I}$ is an ideal in R. □

Exercise 4.
Let $R = \mathbb{C}[x, y]$.

- Compute $\langle x^2, y\rangle \cap \langle x-1, y-1\rangle$, and use this to show $V(\langle x^2, y\rangle \cap \langle x-1, y-1\rangle) = V(x^2, y) \cup V(x-1, y-1)$.

  Hint: Show that the two ideals are coprime and use the chinese remainder theorem.

- Make a sketch of the real part of the variety of $I_1 = \langle y - x^2 + x + 4\rangle$ and of the variety of $I_2 = \langle x-2\rangle$. Then compute $V(I_1 + I_2)$ and verify that $V(I_1 + I_2)$ is contained in the varieties of I_1 and I_2.
  Repeat this for $J_1 = \langle y - x^3 + 3x^2 + x - 3\rangle$, $J_2 = \langle -y - x + 3\rangle$.

Proof:

- We show first that $I := \langle x^2, y\rangle$ and $J := \langle x-1, y-1\rangle$ are coprime ideals in R: Since $y \in I$ and $y-1 \in J$ it follows that $-1 = -y + y - 1 \in I + J$ and thus $R = I + J$. So the ideals are coprime. By the chinese remainder theorem we know that $I \cap J = I \cdot J$.
  So $\langle x^2, y\rangle \cap \langle x-1, y-1\rangle = \langle x^3 - x^2, x^2y - x^2, xy - y, y^2 - y\rangle$.
  Next we compute the varieties $V(\langle x^2, y\rangle \cap \langle x-1, y-1\rangle)$ and $V(x^2, y)$ and $V(x-1, y-1)$. By definition this is the set of elements in $\mathbb{C}^2$ for which all polynomials in the given set vanish. So $V(x-1, y-1) = \{(1,1)\}$, $V(x^2, y) = \{(0,0)\}$.
  Also by considering the generators we see that $V(\langle x^2, y\rangle \cap \langle x-1, y-1\rangle) = V(\langle x^3 - x^2, x^2y - x^2, xy - y, y^2 - y\rangle) = \{(0,0), (1,1)\}$.

- By Lemma 1.1.9 we know that $V(I_1 + I_2) = V(I_1) \cap V(I_2)$, so $V(I_1 + I_2) = V(y - x^2 + x + 4) \cap V(x-2)$. We know $V(x-2) = \{(2, b) | b \in \mathbb{C}\}$. So to compute the intersection we can simply set $x = 2$ into $y - x^2 + x + 4$ and compute its zero set: $y = 2^2 - 2 - 4 = -2$ so $V(I_1 + I_2) = \{(2, -2)\}$. It is easy to verify that this is contained in the varieties $V(I_1)$ and $V(I_2)$: for $x = 2, y = -2$ we have $x - 2 = 0$ and $y - x^2 + x + 4 = 0$.

  By Lemma 1.1.9 we know that $V(J_1 + J_2) = V(J_1) \cap V(J_2)$, so $V(J_1 + J_2) = V(y - x^3 + 3x^2 + x - 3) \cap V(-y - x + 3)$.
  To compute the intersection we can set $y = 3 - x$ into $y - x^3 + 3x^2 + x - 3$ and compute its zero set. So we need to solve for x: $3 - x = x^3 - 3x^2 - x + 3 \Leftrightarrow 0 = x^3 - 3x^2$. So $V(J_1 + J_2) = \{(0,3), (3,0)\}$.

  It is easy to verify that this is contained in the varieties $V(J_1)$ and $V(J_2)$: for $x = 0, y = 3$ we have $y - x^3 + 3x^2 + x - 3 = 0$ and $-y - x + 3 = 0$ as well as for $x = 3, y = 0$.

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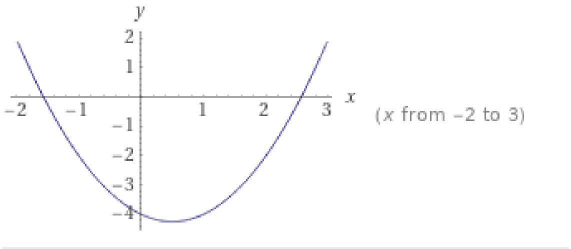

Abbildung 1: $y = x^2 - x - 4$

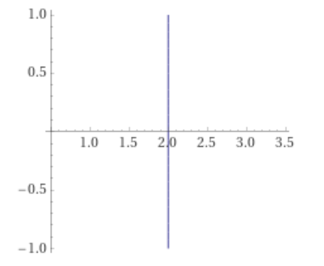

Abbildung 2: $x - 2 = 0$

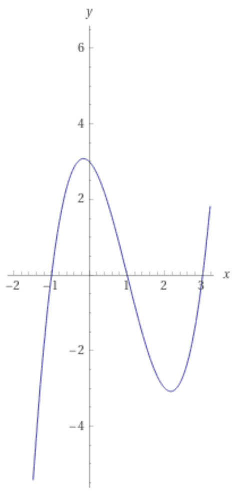

Abbildung 3: $y - x^3 + 3x^2 + x - 3 = 0$

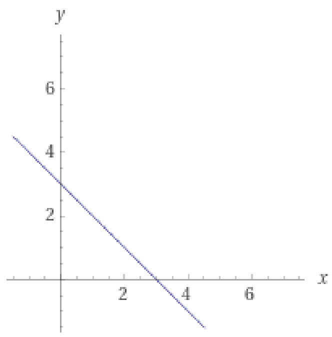

Abbildung 4: $-y - x + 3 = 0$