Eberhard-Karls-Universität Tübingen
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4. November 2020

Introduction to Commutative Algebra and Algebraic Geometry
Exercise Sheet 1

Exercise 1.
Let R be a commutative ring with one. For a subset $S \subset R$ the ideal generated by S is defined as the smallest ideal of R containing S:

$$\langle S \rangle := \bigcap_{\substack{J \textbf{ ideal of } R, \\ S \subset J}} J.$$

Show:

$$\langle S \rangle = \{\sum_{i=1}^{n} \lambda_i s_i \mid n \in \mathbb{N},\, \lambda_i \in R,\, s_i \in S \ \forall i \in \{1, ..., n\}\}.$$

Exercise 2.
Let K be a field and let $R = K[x]$. Let $I = aR$, $J = bR \subset K[x]$ be two principal ideals. Show

$$I + J = \langle \mathsf{gcd}(a, b) \rangle$$

$$I \cap J = \langle \mathsf{lcm}(a, b) \rangle,$$

where gcd denotes the greatest common divisor and lcm the least common multiple.

Exercise 3.
Let R be a commutative ring with one. Let $I \subset R$ be an ideal. Show that $\sqrt{I}$ is an ideal in R.

Exercise 4.
Let $R = \mathbb{C}[x, y]$.

- Compute $\langle x^2, y \rangle \cap \langle x - 1, y - 1 \rangle$, and use this to show $V(\langle x^2, y \rangle \cap \langle x - 1, y - 1 \rangle) = V(x^2, y) \cup V(x - 1, y - 1)$.

  Hint: Show that the two ideals are coprime and use the chinese remainder theorem.

- Make a sketch of the real part of the variety of $I_1 = \langle y - x^2 + x + 4 \rangle$ and of the variety of $I_2 = \langle x - 2 \rangle$. Then compute $V(I_1 + I_2)$ and verify that $V(I_1 + I_2)$ is contained in the varieties of I_1 and I_2.
  Repeat this for $I_1 = \langle y - x^3 + 3x^2 + x - 3 \rangle$, $I_2 = \langle -y - x + 3 \rangle$.

Submission: In groups of up to three students, until Wednesday, 11. November 2020, 12:00 o'clock via URM. You are allowed to submit your solutions in German.