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Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 10

Exercise 1.
Let K be an algebraically closed field and let X be an affine variety. Let $f : X \to K$ be a map. Prove that the following statements are equivalent:

(i) $f \in \mathcal{O}_X(X)$.

(ii) $f : X \to K$ is a morphism.

Proof: „(i) $\Rightarrow$ (ii)": We have to show that $\forall V \subset K, h \in \mathcal{O}_K(V)$ we have $h \circ f \in \mathcal{O}_X(f^{-1}(V))$. It suffices to prove this for basis open sets $V = D(g)$ with $g \in K[t]$.
We prove that for $V = D(g)$ we have $\mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_X(D(g \circ f))$. This holds because

$$f(x) \in V(g) \Leftrightarrow g \circ f = 0$$

and so

$$f^{-1}(D(g)) = \{x \in X \mid g \circ f \neq 0\} = D(g \circ f).$$

So $\mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_X(D(g \circ f)) = K[X]_{g\circ f}$.
For $V = D(g)$ we have $\mathcal{O}_K(V) = K[t]_g$ and any $h \in \mathcal{O}_K(V)$ is of the form $h = \frac{h'}{g^r}$ for some $h' \in K[t], r \in \mathbb{N}$.
Also since $f \in \mathcal{O}_X(X) = K[X]$ and $g, h \in K[t]$ we have $h \circ f, g \circ f \in K[X]$ (follows from the definition).
Combined it follows that for any $h \in \mathcal{O}_K(V)$ we have $h \circ f = \frac{h' \circ f}{(g \circ f)^r} \in K[X]_{g\circ f} = \mathcal{O}_X(f^{-1}(V))$.

„(ii) $\Rightarrow$ (i)": Since f is a morphism, we know that for all open sets $V \subset K$, $g \in \mathcal{O}_K(V)$ we have $g \circ f \in \mathcal{O}_X(f^{-1}(V))$. Choosing $V = K$ and $g = id$ we obtain $f = id \circ f \in \mathcal{O}_X(f^{-1}(K)) = \mathcal{O}_X(X)$. □

Exercise 2.
Let X be an affine variety and $\mathcal{F}$ its sheaf of regular functions. Let $U \subset X$ be an open set.
Let $s \in \mathcal{F}(U)$ be an element with $s_x = 0 \in \mathcal{F}_x$ for all $x \in U$. Show: $s = 0$.

Proof: We fix $x \in U$. Since $s_x \in \mathcal{F}_x$ is an equivalence class, $s_x = 0$ is an equation of equivalence classes in $\mathcal{F}_x$. This means that there exists two open neighbourhoods of $x \in U$, say U' and U'' such that $s \in \mathcal{F}(U')$ and $0 \in \mathcal{F}(U'')$ and such that there exists an open neighbourhood $x \in V \subset U' \cap U''$ with $s_{|V} = 0_{|V}$. This implies in particular that $s(x) = 0$.
We had $x \in U$ fixed, but since $s_x = 0 \in \mathcal{F}_x$ for all $x \in U$ it follows that $s_{|U} = 0$ and since s is a regular function on U ($s \in \mathcal{F}(U)$), we have $s = s_{|U} = 0$. □

Exercise 3.
Let K be an algebraically closed field with char$(K) = 0$. Consider the map from the affine line $\mathbb{A}^1_K$ to the curve $C = V(y^2 - x^3)$ given by $\phi : \mathbb{A}^1_K \to C, t \mapsto (t^2, t^3)$. Prove ϕ is a morphism and a homeomorphism (i.e. bijective, continuous and open).

Solution: We have $\phi : \mathbb{A}^1_K \to C, t \mapsto (t^2, t^3)$. Since ϕ is polynomial, it follows that ϕ is a morphism (3.3.5) and continuous (1.6.2). Furthermore, ϕ is surjective, since for any point $p = (a, b) \in C$ we can choose $t = \sqrt{a}$. Then $(t^2, t^3) = (a, \sqrt{a}^3) = (a, b)$ since $b^2 = a^3$ so $b = \sqrt{a^3} = \sqrt{a}^3$.
Also ϕ is injective, since for $(t^2, t^3) = (q^2, q^3)$ we have $t = \pm q$ (since $t^2 = q^2$) and if $t = -q$ it follows that $t^3 = -q^3 \neq q^3$ except for $q = 0$. So $(t^2, t^3) = (q^2, q^3)$ implies $t = q$.
It remains to show that ϕ is open. Equivalently to proving ϕ is open, we can show that ϕ is closed, i.e. that for closed subsets $V \subset K = \mathbb{A}^1_K$ the image $\phi(V)$ is closed in C. The closed sets in K are K, $\emptyset$ and any finite number of points. For these the images under ϕ are $\phi(K) = C$, $\phi(\emptyset) = \emptyset$ and $\phi(\{a_1, \dots, a_n\}) = \{\phi(a_1), \dots, \phi(a_n)\}$. Since C and $\emptyset$ are closed in the subspace topology $C \subset K^2$, it remains to consider the sets of finitely many points. In K^2 any finite number of points is a closed subset, so sets of finitely many points in C are closed in C.

□