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Wintersemester 2020/2021
3. February 2021

Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 11

Exercise 1.
For a ring extension $R' \supset R$, we say $\alpha \in R'$ is *integral* over R, if there exists $f \in R[x] \setminus \{0\}$ with $f(\alpha) = 0$ and $\mathsf{LC}(f) = 1$.
R' is called *integral* over R if $\forall \alpha \in R'$: α is integral over R.

Let K be a field. Let R be a finitely generated K-algebra without zero divisors and let $\beta_1, \ldots, \beta_d \in R \subset \mathsf{Quot}(R)$.
Prove: If R is integral over $K[\beta_1, \ldots, \beta_d]$, then $\mathsf{Quot}(R)$ is algebraic over $K(\beta_1, \ldots, \beta_d)$.

Solution: Let $\beta_1, \ldots, \beta_d \in R$ be algebraically independent over K and let R be integral over $K[\beta_1, \ldots, \beta_d]$.
Let $r \in R \setminus \{0\}$ be an arbitrary element. Since R is integral over $K[\beta_1, \ldots, \beta_d]$, there exists $f = \sum_{i=0}^{m} c_i x^i \in K[\beta_1, \ldots, \beta_d][x] \setminus \{0\}$ such that $f(r) = 0$, $c_m = 1$.
Define $g := \sum_{i=0}^{m} c_{m-i} x^i \in K[\beta_1, \ldots, \beta_d][x]$. Then $g \neq 0$ and

$$\begin{aligned} r^m \cdot g(\frac{1}{r}) &= r^m \sum_{i=0}^{m} c_{m-i} (\frac{1}{r})^i \\ &= \sum_{i=0}^{m} c_{m-i} r^{m-i} \\ &= \sum_{i=0}^{m} c_i r^i \\ &= f(r) \\ &= 0 \end{aligned}$$

Since R contains no zero divisors, it follows that $g(\frac{1}{r}) = 0$.
$\Rightarrow \frac{1}{r}$ is algebraic over $K(\beta_1, \ldots, \beta_d)$.
$\Rightarrow \mathsf{Quot}(R)$ is algebraic over $K(\beta_1, \ldots, \beta_d)$. $\square$

Exercise 2 (Semicontinuity of the dimension)**.**
Let X be an affine variety with irreducible components $X_1, \ldots, X_r$. For $x \in X$ we set

$$\dim(X_x) := \max\{\dim(X_i) \,:\, 1 \leq i \leq r,\, x \in X_i\}$$

Prove: The function $X \to Z$, $x \mapsto \dim(X_x)$ is upper semicontinuous, i.e., $\forall x \in X$ there is an open neighbourhood $U \subset X$ with $\dim(X_u) \leq \dim(X_x)$ for all $u \in U$.

Proof: Write $X = X_1 \cup \ldots \cup X_r$. Let $x \in X$ be arbitrary. Then there exist $i_1, ..., i_k$ such that $x \in X_{i_j}$, $j \in \{1, ..., k\}$ and $x \notin X_i$ with $i \notin \{i_1, ..., i_k\}$.
Let X_{i_1} be the irreducible component of the largest dimension that contains x, so $\dim(X_x) = \dim(X_{i_1})$.
In case that there exists $i \notin \{i_1, ..., i_k\}$, we choose $U = X_{i_1} \setminus (\bigcup_{i \in \{1,...,r\} \setminus \{i_1,...,i_k\}} X_i$. Then $x \in U$ and U is open. Moreover, for all elements $u \in U$ we have $\dim(X_u) = \dim(X_{i_1}) = \dim(X_x)$.
If otherwise $\{i_1, ..., i_k\} = \{1, ..., r\}$, then we choose any $U \subset X$ open with $x \in U$ and we obtain: $\dim(X_u) = \dim(X_{i_1}) = \dim(X_x)$, since X_{i_1} was the irreducible component of the largest dimension that contains x. $\square$

Exercise 3.

a) Determine the field of rational functions and the dimension of the cuspidal cubic $X = V(x^2 - y^3) \subset K^2$.

Solution: We know from Exercise Sheet 3, exercise 2, that the coordinate ring of X is

$$K[X] = K[x, y]/\langle x^2 - y^3 \rangle \cong K[t^2, t^3].$$

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Now $t = \frac{t^3}{t^2} \in \mathsf{Quot}(K[t^2, t^3])$, so $\mathsf{Quot}(K[t^2, t^3]) = \mathsf{Quot}(K[t]) = K(t)$.
Therefore the dimension of the cuspidal cubic is

$$\dim(X) = \mathsf{trdeg}_K(\mathsf{Quot}(K[t^2, t^3])) = \mathsf{trdeg}_K(\mathsf{Quot}(K(t))) = 1.$$

□

b) Let U be a d-dimensional vector subspace of K^n. Determine the dimension of U as an affine variety in K^n and compare it to the dimension of U as a K vector space.

Solution: Let $v_1, \ldots, v_d$ be a basis of $U \subset K^n$ as K-vector space. We extend this to a basis of K^n $v_1, .., v_n$. With the change of coordinates $K^n \to K^n$ via $(e_1, ..., e_n) \to (v_1, .., v_d, v_{d+1}, ..., v_n)$ we can write U as the image of $\begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & 0 \end{pmatrix}$.
This map, which is an isomorphism of affine varieties, since it is linear, together with $K[K^n] = K[x_1, ..., x_n]$ implies that $I(U) = \langle x_{d+1}, ..., x_n \rangle$. Then $K[U] = K[x_1, ..., x_n]/I(U) \cong K[x_1, ..., x_d]$.
So $\dim(U) = \mathsf{trdeg}_K(K(x_1, ..., x_d)) = d$.
This is the same as the K-vector space dimension of U. □