Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

Wintersemester 2020/2021
27. January 2021

Introduction to Commutative Algebra and Algebraic Geometry
Exercise Sheet 11

Exercise 1.
For a ring extension $R' \supset R$, we say $\alpha \in R'$ is *integral* over R, if there exists $f \in R[x] \setminus \{0\}$ with $f(\alpha) = 0$ and $\mathsf{LC}(f) = 1$.
R' is called *integral* over R if $\forall \alpha \in R'$: α is integral over R.

Let K be a field. Let R be a finitely generated K-algebra without zero divisors and let $\beta_1, \ldots, \beta_d \in R \subset \mathsf{Quot}(R)$.
Prove: If R is integral over $K[\beta_1, \ldots, \beta_d]$, then $\mathsf{Quot}(R)$ is algebraic over $K(\beta_1, \ldots, \beta_d)$.

Exercise 2 (Semicontinuity of the dimension)**.**
Let X be an affine variety with irreducible components $X_1, \ldots, X_r$. For $x \in X$ we set

$$\dim(X_x) := \max\{\dim(X_i) \,:\, 1 \leq i \leq r,\, x \in X_i\}$$

Prove: The function $X \to \mathbb{Z}$, $x \mapsto \dim(X_x)$ is upper semicontinuous, i.e., $\forall x \in X$ there is an open neighbourhood $U \subset X$ with $\dim(X_u) \leq \dim(X_x)$ for all $u \in U$.

Exercise 3.
Let K be an algebraically closed field.

a) Determine the field of rational functions and the dimension of the cuspidal cubic $X = V(x^2 - y^3) \subset K^2$.

b) Let U be a d-dimensional vector subspace of K^n. Determine the dimension of U as an affine variety in K^n and compare it to the dimension of U as a K vector space.

Submission: In groups of up to three students, until Wednesday, 3. February 2021, 12:00 o'clock via URM. You are allowed to submit your solutions in German.