Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

Wintersemester 2020/2021
10. February 2021

Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 12

Exercise 1.
Let K bei an algebraically closed field, $f, g \in K[x_1, \ldots, x_n]$, $V(f,g) \subset K^n$. Prove the equivalence of the following statements:

(i) $\gcd(f,g) = 1$.

(ii) $\dim(V(f,g)) \leq n-2$.

Proof: (i)⇒ (ii): Assume $\dim(V(f,g)) > n-2$. If $\dim(V(f,g)) = n$, then $f = g = 0$ (otherwise $V(f,g) \subset V(f)$ where w.l.o.g f not constant and thus $\dim(V(f,g)) \leq n-1$). It follows $gcd(f,g) = gcd(0,0) = 0 \neq 1$.
If $\dim(V(f,g)) = n-1$, either both f and g have to be not-constant or one of f and g is zero and the other not-constant. In the case one of f and $g = 0$, we can assume w.l.o.g. $g = 0$. Then we have $V(f,g) = V(f)$ and $gcd(f,0) = f \neq 1$, since f is not constant. Otherweise, f, g are both not constant. Then $V(f,g)$ it is a union of irreducible components and with at least one of these components of dimension $n-1$. This component is then a hypersurface $V(p)$ by by 4.1.13 with p non-constant. So $V(p) \subset V(f,g)$. It follows that $f, g \in \langle p \rangle$ so p divides the gcd of f and g: $\gcd(f,g) \neq 1$.

(ii)⇒ (i): We assume $gcd(f,g) \neq 1$. It follows that either $f = g = 0$ or f and g are bot not constant. If $f = g = 0$ it follows that $\dim(V(f,g)) = \dim(K^n) = n$. If both f, g are not constant, we assume $p = \gcd(f,g)$ is not constant. Then $\langle f, g \rangle \subset \langle p \rangle$ so $V(f,g) \supset V(p)$ and $V(p)$ is a hpyersurface and thus of dimension $n-1$ by 4.1.12. So $\dim(V(f,g)) \geq n-1$. □

Exercise 2.
Show that the statement from Theorem 4.2.1 2) does not generally hold if K is not algebraically closed.
For this consider $K = \mathbb{R}$ and $X = Y = K$ and $\varphi : X \to Y$, $t \mapsto t^2$ and the ideal $J = \langle x^2 + 1 \rangle \subset K[x]$.

Solution: Recall the statement from Theorem 4.2.1 2): Let $\varphi : X \to Y$ be a morphism of algebraic varieties, $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$. If $J \subset \mathcal{O}_X(X)$ is an ideal and $A = V(J) \subset X$, then $\overline{\phi(A)} = V((\varphi^*)^{-1}(J)) \subset Y$, $I(\varphi(A)) = (\varphi^*)^{-1}(\sqrt{J})$.

Now let $K = \mathbb{R}$ and $X = Y = K$, $\varphi : X \to Y$, $t \mapsto t^2$ and the ideal $J = \langle x^2 + 1 \rangle \subset K[x]$. First we need to compute $\overline{\phi(V(J))}$ and $V((\varphi^*)^{-1}(J))$.
Since $K = \mathbb{R}$ we have $V(x^2 + 1) = \emptyset$. So $\overline{\phi(V(J))} = \emptyset$.
We know that $\varphi^* : K[y] \to K[x]$, $y \mapsto x^2$. So $V((\varphi^*)^{-1}(J)) = V(\langle y + 1 \rangle)$.
Now $\emptyset \neq V(\langle y + 1 \rangle)$.
Further we need to compute $I(\varphi(V(J)))$ and $(\varphi^*)^{-1}(\sqrt{J})$.
We have $\varphi(V(J)) = \varphi(\emptyset) = \emptyset$. Now $I(\emptyset)$ are all those polynomials in $\mathbb{R}[y]$ that vanish everywhere in $\emptyset$. So $I(\emptyset) = \mathbb{R}[y]$.
Since $J = \langle x^2 + 1 \rangle$ is a prime ideal over $\mathbb{R}[x]$ it is radical, so $(\varphi^*)^{-1}(\sqrt{J}) = \langle y + 1 \rangle$.
We see $\mathbb{R}[y] \neq \langle y + 1 \rangle$.

So the statement of Theorem 4.2.1 2) does in general not hold, if K is not algebraically closed. □

Exercise 3.
Determine the number of irreducible components, the dimension and the ring of regular functions for every fibre of the following morphisms:

(i) $\varphi : K^2 \to K, (z, w) \mapsto zw$,

(ii) $\psi : K^2 \to K^2, (z, w) \mapsto (zw, w)$.

Solution:

Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

Wintersemester 2020/2021
10. February 2021

Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 12

(i) Let $p \in K$. Then $\varphi^{-1}(p) = \{(a,b) \in K^2 | ab = p\} = V(xy - p)$.
If $p = 0$, $V(xy-p) = V(xy) = V(x) \cup V(y)$ are the two coordinate axes, so there are two irreducible components. Each of dimension 1. The ring of regular functions for $V(xy)$ is by Corollary 4.2.2 $K[x,y]/\sqrt{\varphi^*(m_0)}$. Now $\varphi^* : K[t] \to K[x,y]$, $t \mapsto xy$, so $\varphi^*(m_0) = \varphi^*(\langle t \rangle) = \langle xy \rangle$. Since $\sqrt{xy} = \langle xy \rangle$ we have $K[x,y]/\sqrt{\varphi^*(m_0)} = K[x,y]/\langle xy \rangle$.
For $p \neq 0$ we have $V(xy-p)$ is an irreducible hyperbolic curve, so there is just one irreducible component. The ring of regular functions in this case is $K[x,y]/\sqrt{\varphi^*(m_p)} \cong K[x, x^{-1}]$, since $\varphi^*(m_p) = \varphi^*(\langle t - p \rangle) = \langle xy - p \rangle$ is radical and $\overline{y} = \frac{\overline{p}}{x}$. Since Quot$(K[x, x^{-1}]) = K(x)$, we see that the dimension of the fibre in this case is 1.

(ii) Let $p = (p_1, p_2) \in K^2$. Then $\psi^{-1}(p) = \{(a,b) \in K^2 | ab = p_1, b = p_2\} = V(xy - p_1, y - p_2)$.
If $p = (0,0)$ we have the fibre $\psi^{-1}(p) = V(xy, y) = V(y)$, so there is only one irreducible component. Further, we have $\psi^* : K[t_1, t_2] \to K[x,y]$, $(t_1, t_2) \mapsto (xy, y)$. So $\psi^*(m_{(0,0)}) = \langle xy, y \rangle = \langle y \rangle$ and by 4.2.2 the ring of regular functions is $K[x,y]/\langle y \rangle = K[x]$. So the dimension is 1.
If $p_1 = 0$, $p_2 \neq 0$, we have the fibre $\psi^{-1}(p) = V(xy, y - p_2) = V(x, y - p_2)$. This is exactly one point, so there is only one irreducible component. The ring of regular functions is in this case given as: $K[x,y]/\langle x, y - p_2 \rangle = K$. Since $\psi^*(m_{0,p_2}) = \langle xy, y - p_2 \rangle = \langle x, y - p_2 \rangle$ and this is a maximal ideal. It follows that the fibre has dimension 0.
Note, that $(a, 0) \notin \text{im}(\psi)$ if $a \neq 0$. So we do not need to consider the case $p_1 \neq 0$, $p_2 = 0$.
If $p_1 \neq 0, p_2 \neq 0$, the fibre is $\psi^{-1}(p) = V(xy - p_1, y - p_2) = \{(\frac{p_1}{p_2}, p_2)\}$. So there is only one irreducible component. The ring of regular functions in this case is $K[x,y]/\langle xy - p_1, y - p_2 \rangle = K$ (explanation analguous to before). This fibre also has dimension 0.