Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

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Introduction to Commutative Algebra and Algebraic Geometry Solution to Exercise Sheet 13

Exercise 1.
Consider $X := Y := K^2$ and the morphism $\varphi : X \to Y$, $(z, w) \mapsto (zw, w)$. Prove:

(i) φ is birational, but not an isomorphism.

(ii) For $g(a, b) = \frac{a}{b}$ we have $g \in K(Y) \setminus \mathcal{O}(Y)$ but $\varphi^*(g) \in \mathcal{O}(X)$.

Proof:

(i) Since $X = Y = K^2$ we have that φ is a map between irreducible affine varieties. Moreover, we know from exercise sheet 12, that $\text{im}(\varphi) = K^2 \setminus \{(a, 0)|a \in K \setminus \{0\}$, so φ is not bijective and hence it cannot be an isomorphism. However, $\overline{\text{im}(\varphi)} = K^2$, so φ is dominant.
Now we consider $\varphi^* : K[x, y] \to K[z, w]$, $x \mapsto zw$, $y \mapsto w$. Then we can extend φ^* to the quotient fields: $\varphi^* : K(x, y) \to K(z, w)$, $x \mapsto zw$, $y \mapsto w$.
Now φ^* is injective, because φ is dominant (4.2.6).
When considering $\psi : K(z, w) \to K(x, y)$, $z \mapsto \frac{x}{y}$, $w \mapsto y$, we see that $\varphi^* \circ \psi = id_{K(z,w)}$, so φ^* is surjective.
We have shown that φ^* is an isomorphism, so φ is birational. □

(ii) We know that the projections $\pi_1 : K^2 \to K, (a, b) \mapsto a$ and $\pi_2 : K^2 \to K, (a, b) \mapsto b$ are regular functions of Y. Thus, $g = \frac{\pi_1}{\pi_2} \in K(Y)$. However, $g \notin \mathcal{O}(Y) = K[K^2] = K[x, y]$.
But $\varphi^* : K(x, y) \to K(z, w)$, $x \mapsto zw$, $y \mapsto w$ so $\varphi^*(g) = z \in \mathcal{O}(X) = K[z, w]$. □

Exercise 2.
Let K be an algebraically closed field and $X = V(x^3 - y^2) \subset K^2$. Prove: the coordinate ring $K[X]$ is not normal.

Proof: We know that $K[C] = K[x, y]/\langle x^3 - y^2\rangle$, since $\langle x^3 - y^2\rangle$ is a prime ideal. Now let $\overline{x}$ and $\overline{y}$ be the cosets of x and $y \in K[C]$ respectively. This gives $\frac{\overline{x}}{\overline{y}} \in K(C)$. Also due to the equivalence relation we have $(\frac{\overline{x}}{\overline{y}})^2 = \overline{x}$. So the polynomial $z^2 - \overline{x} \in K[x, y]/\langle x^3 - y^2\rangle[z]$ and thus $\frac{\overline{y}}{\overline{x}}$ is integral over $K[C]$. Consequently, $K[C]$ and hence C are not normal.

Alternatively: We know from exercise sheet 3 that $K[C] = K[x, y]/\langle x^3 - y^2\rangle \cong K[T^2, T^3]$. Thus, $K(C) \cong K(T)$. We know that $T \in K(T)$. Moreover $x^2 - T^2 \in K[T^2, T^3]$ is a polynomial over $K[T^2, T^3]$ with leading coefficient 1 that annulates T. So T is integral over $K[T^2, T^3]$. Hence, $K[T^2, T^3]$ is not normal. Since being normal is a property of a ring that is maintained by ringisomporphisms, $K[C]$ and hence C are not normal. □

Exercise 3.
Let $R \subset S$ and $S \subset T$ be integral ring extensions. Prove that $R \subset T$ is an integral ring extension.

Proof: Let $t \in T$ be an arbitrary element. Since $S \subset T$ is an integral ring extension, there exist $s_0, \ldots, s_{n-1} \in S$ such that $t^n + s_{n-1}t^{n-1} + \ldots + s_0 = 0$. We consider the ring extension $R[s_0, \ldots, s_{n-1}][t] \supset R[s_0, \ldots, s_{n-1}] \supset R$. Since $s_0, \ldots, s_{n-1}$ are integral over R, it follows by 4.3.3. that $R[s_0, \ldots, s_{n-1}] \supset R$ is a finitely generated R module. Since t is integral over $R[s_0, \ldots, s_{n-1}]$, it follows again by 4.3.3, that $R[s_0, \ldots, s_{n-1}][t] \supset R[s_0, \ldots, s_{n-1}]$ is an integral ring extension and that $R[s_0, \ldots, s_{n-1}][t]$ is a finitely generated $R[s_0, \ldots, s_{n-1}]$-module.
Let $\alpha_1, ..., \alpha_k$ be a generating set for $R[s_0, \ldots, s_{n-1}][t]$ as a finitely generated $R[s_0, \ldots, s_{n-1}]$-module and let $\beta_1, ..., \beta_m$ be a generating set for $R[s_0, \ldots, s_{n-1}] \supset R$ as a finitely generated R module.
Then $\alpha_i \cdot \beta_j$, $i = 1, ..., k$, $j = 1, ..., m$ is a finite generating set for $R[s_0, \ldots, s_{n-1}][t]$ as R-module.
So $R[s_0, \ldots, s_{n-1}][t]$ is a finitely generated R-module and thus by 4.3.3 t is integral over R. □