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Introduction to Commutative Algebra and Algebraic Geometry Exercise Sheet 2 - Solution

Exercise 1.
Let K be a field. Consider $R := K[T^2, T^3] \subset K[T]$ and prove the following statements:

1. R is noetherian.
2. $\sqrt{T^2} = \langle T^2, T^3 \rangle$.

Proof:

1. Since K is a field, K is noetherian. Also $K[T^2, T^3]$ is a finitely generated K-algebra. With Corollary 1.4.5 it follows that $K[T^2, T^3]$ is noetherian. □

2. We know $\sqrt{T^2} = \{p \in K[T^2, T^3] | \, \exists n \in \mathbb{N} : p^n \in \langle T^2 \rangle\}$. With $n = 1$ we know that $T^2 \in \sqrt{T^2}$ and with $n = 2$ it follows that $T^3 \in \sqrt{T^2}$. So $\langle T^2, T^3 \rangle \subset \sqrt{T^2}$. It remains to show that $\sqrt{T^2} \subset \langle T^2, T^3 \rangle$. We do this by proving that $\langle T^2, T^3 \rangle$ is a maximal ideal and $\sqrt{T^2} \neq R$.
We consider the quotient ring:
$$R/\langle T^2, T^3 \rangle = K[T^2, T^3]/\langle T^2, T^3 \rangle \cong K.$$
So $\langle T^2, T^3 \rangle$ is a maximal ideal in R. Since $1 \notin \langle T^2 \rangle$ and $1^n = 1 \forall n \in \mathbb{N}$ it follows that $1 \notin \sqrt{T^2}$ so $\sqrt{T^2} \neq R$. Therefore, as $\sqrt{T^2}$ contains the maximal ideal $\langle T^2, T^3 \rangle$ they have to be equal. □

Exercise 2.
Let (X, Ω) be a topological space. The closure of a subset $A \subset X$ is the intersection $\overline{A}$ of all closed subsets $B \subset X$ with $A \subset B$. Show:

1. A subset $A \subset X$ is closed in X if and only if $A = \overline{A}$.
2. For every finite union $A := A_1 \cup \ldots \cup A_n$ of subsets $A_1, \ldots, A_n \subset X$ the following applies: $\overline{A} = \overline{A_1} \cup \ldots \cup \overline{A_n}$.
3. Let $A \subset B \subset X$ be subsets. The closure of A in B with respect to the subspace topology is given by $\overline{A} \cap B$. The *subspace topology* of $B \subset X$ is given by the following system of open sets: $\Omega_B = \{Y \cap B | Y \subset X \text{open}\}$.

Proof:

1. „⇒“: Let $A \subset X$ be closed. Since $\overline{A} = \bigcap_{\substack{B \subset X \textbf{closed}, \\ A \subset B}} B$, we know that $\overline{A} \subset A$ (because $A \subset A$ closed in X). Also since $\overline{A}$ is the intersection over sets of X which contain A it follows that $A \subset \overline{A}$.
„⇐“: Let $A \subset X$ such that $A = \overline{A}$. Since any intersection of closed subsets of a topological space is closed and $\overline{A}$ is such an intersection, $\overline{A}$ is closed. So A is closed in X. □

2. Let $A := A_1 \cup \ldots \cup A_n$ be a finite union of subsets $A_1, \ldots, A_n \subset X$. We prove $\overline{A} = \overline{A_1} \cup \ldots \cup \overline{A_n}$ inductively. For $n = 1$ there is nothing to show. Let $n = 2$. We need to show $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.
Since $\overline{A_1} \cup \overline{A_2}$ is closed and contains $A_1 \cup A_2$, it also contains $\overline{A_1 \cup A_2}$. Moreover, $\overline{A_1 \cup A_2}$ is closed and contains A_1, so it also contains $\overline{A_1}$. Since the same is true for A_2 it also contains $\overline{A_2}$ and thus $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.
For the induction step we assume that the statement is true for $n - 1$. By using the statement for $n = 2$ and the induction assumption the claim follows:
$$\begin{aligned}\overline{(A_1 \cup \ldots A_{n-1}) \cup A_n} &= \overline{(A_1 \cup \ldots A_{n-1})} \cup \overline{A_n} \\ &= \overline{A_1} \cup \ldots \cup \overline{A_{n-1}} \cup \overline{A_n}\end{aligned}$$
□

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3. Let $A \subset B \subset X$ be subsets. The closure of A in B is the intersection of all closed subsets of B which contain A.

$$\overline{A}^B = \bigcap_{\substack{Y \subset B \text{ closed} \\ A \subset Y}} Y$$

A set $Y \subset B$ is closed in B if and only if there exists $Y' \subset X$ closed such that $Y = Y' \cap B$. Now if $A \subset Y$ then $A \subset Y'$ for $Y = Y' \cap B$. So we can write:

$$\begin{aligned}\overline{A}^B &= \bigcap_{\substack{Y = Y' \cap B \subset B, Y' \subset X \text{ closed} \\ A \subset Y}} Y \\ &= \bigcap_{\substack{Y' \subset X \text{ closed} \\ A \subset Y'}} Y' \cap B \\ &= \overline{A} \cap B\end{aligned}$$

□

Exercise 3. 1. Let $V_1 = \{1,3,5\} \subset \mathbb{R}$ and $V_2 = \{1,2,3,4,5\} \subset \mathbb{R}$. Compute $I(V_1) \subset \mathbb{R}[x]$ and $I(V_2) \subset \mathbb{R}[x]$. Prove $I(V_2) \subset I(V_1)$.

2. Let $V_3 = \{(0,0),(0,2),(1,0),(1,1),(1,-2)\}, V_4 = \{(1,-1),(1,0),(0,0),(3,1),(0,2)\} \subset \mathbb{R}^2$. Compute $I(V_3), I(V_4) \subset \mathbb{R}[x,y]$ and $I(V_3 \cap V_4)$.

3. Let $M \neq \emptyset$ be an arbitrary finite subset of $\mathbb{R}^2$. Find a polynomial $f \in \mathbb{R}[x,y]$ such that $M = V(f)$.

4. Consider $\mathbb{Z} \subset \mathbb{R}$. Compute $I(\mathbb{Z}) \subset \mathbb{R}[x]$.

Solution:

1. We compute:

$$\begin{aligned}I(V_1) &= \{f \in \mathbb{R}[x] | f(1) = 0, f(3) = 0, f(5) = 0\} \\ &= \{f \in \mathbb{R}[x] | (x-1)|f \wedge (x-3)|f \wedge (x-5)|f\} \\ &= \langle (x-1)(x-3)(x-5) \rangle \subset \mathbb{R}[x]\end{aligned}$$

$$\begin{aligned}I(V_2) &= \{f \in \mathbb{R}[x] | f(1) = 0, f(2) = 0, f(3) = 0, f(4) = 0, f(5) = 0\} \\ &= \{f \in \mathbb{R}[x] | (x-1)|f \wedge (x-2)|f (x-3)|f \wedge (x-4)|f \wedge (x-5)|f\} \\ &= \langle (x-1)(x-2)(x-3)(x-4)(x-5) \rangle \subset \mathbb{R}[x]\end{aligned}$$

Since the generator $(x-1)(x-2)(x-3)(x-4)(x-5)$ of $I(V_2)$ is contained in $I(V_1) = \langle (x-1)(x-3)(x-5) \rangle$ we have $I(V_2) \subset I(V_1)$. □

2. Compute:

$$\begin{aligned}I(V_3) &= I(\{(0,0),(0,2),(1,0),(1,1)\}) \\ &= I(\{(0,0)\} \cup \{(0,2)\} \cup \{(1,0)\} \cup \{(1,1)\}) \\ &= I((0,0)) \cap I((0,2)) \cap I((1,0)) \cap I((1,1)) \\ &= \langle x, y \rangle \cap \langle x, y-2 \rangle \cap \langle x-1, y \rangle \cap \langle x-1, y-1 \rangle \\ &= (\langle x, y \rangle \cap \langle x, y-2 \rangle) \cap (\langle x-1, y \rangle \cap \langle x-1, y-1 \rangle) \\ &= \langle x^2, xy-2x, xy, y^2-2y \rangle \cap \langle (x-1)^2, (x-1)(y-1), xy-y, y^2-y \rangle \\ &= \langle x^2, xy-2x, xy, y^2-2y \rangle \cap \langle (x-1)^2, (x-1)(y-1), (x-1)y, y^2-y \rangle \\ &= \langle x, y^2-2y \rangle \cap \langle (x-1), y^2-y \rangle \\ &= \langle x(x-1), x(y^2-y), (y^2-2y)(x-1), (y^2-2y)(y^2-y) \rangle\end{aligned}$$

Here we used Lemma 1.3.5 (6) and the fact that the ideals are coprime. Also we simplified the set of generators by removing superfluous polynomials.
We compute with a different approach: The first polynomial sets all possible values for the x-coordinate. With the following polynomials we make the corresponding values of the y-coordinate explicit. Example: if $x = 0$ it follows that $(x-1)(x-3) \neq 0$ so the remaining factor of the polynomial has to be zero and we can use this to take the values of the y-coorindates belonging to $x = 0$. For $x = 0$ there are two points in V_4 namely: $(0, 0)$ and $(0, 2)$. So the remaining factor for the second polynomial is $y(y - 2)$.

$$\begin{aligned} I(V_4) &= I(\{(1,-1),(1,0),(0,0),(3,1),(0,2)\}) \\ &= \langle x(x-1)(x-3),(x-1)(x-3)y(y-2),x(x-1)(y-1),x(x-3)(y+1)y\rangle \end{aligned}$$

Now we compute $I(V_3 \cap V_4)$:

$$\begin{aligned} I(V_3 \cap V_4) &= I(\{(1,0),(0,0),(0,2)\}) \\ &= \langle x(x-1),(x-1)y(y-2),xy\rangle \supset I(V_3), I(V_4). \end{aligned}$$

□

3. Let $M = \{(a_1, b_1), \dots, (a_n, b_n)\} \subset \mathbb{R}^2$ be an arbitrary non-empty finite subset of $\mathbb{R}^2$. We want to construct a polynomial $f \in \mathbb{R}[x, y]$ such that $M = V(f)$.
We start by thinking about a single point $(a, b) \in \mathbb{R}^2$. We start with the two polynomials $x - a$ and $y - b$ which both have to be zero to give us the required vanishing point. The polynomial $(x - a) + (y - b)$ is a good starting point since $(a, b) \in V((x - a) + (y - b))$. But also $(b, a) \in V((x - a) + (y - b))$, so we have not finished yet. Since we are working over the real numbers we can use that squares are always non-negative and try $(x - a)^2 + (y - b)^2$. That way we prevent cancellations as before and each summand has to be zero in order for the polynomial to be zero. So $V((x - a)^2 + (y - b)^2) = \{(a, b)\}$.
Now for our finite set of arbitrary points we can just take the product of the polynomials we obtain for each of the single points:
$$M = V(\prod_{i=1}^{n}((x-a_i)^2 + (y-b_i)^2))$$
where $\prod_{i=1}^{n}((x - a_i)^2 + (y - b_i)^2) \in \mathbb{R}[x, y]$. □.

4. Consider $\mathbb{Z} \subset \mathbb{R}$. We want $I(\mathbb{Z}) \subset \mathbb{R}[x]$.
Since $\mathbb{Z}$ is not finite we cannot use the same trick as in the previous exercise: For an infinite number of points we will not obtain a polynomial but a power series.
By Lemma 1.3.5 we have $I(A) = I(\overline{A})$. So we can also consider the closure of $\mathbb{Z}$ in $\mathbb{R}$. The closure of $\mathbb{Z}$ is an affine variety in $\mathbb{R}$, i.e., the zero set of some polynomials in $\mathbb{R}[x]$. Any polynomial in $R[x]$ of degree at least one can always only have finitely many zeroes, but $\mathbb{Z}$ is infinite. So $I(\mathbb{Z})$ does not contain polynomials of degree one or higher. However, $I(\mathbb{Z})$ is not empty since, $0 \in I(\mathbb{Z})$. Since non-zero constant polynomials have no zeroes at all, it follows that $I(\mathbb{Z}) = 0$. Furthermore we have shown: $\overline{Z} = \mathbb{R}$ in the Zariski topology. □

Exercise 4.
Let A, B, C be finite groups and let
$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$
be a short exact sequence of group homomorphisms, i.e., α is injective, β is surjective and $\ker(\beta) = \mathrm{im}(\alpha)$. Prove $|B| = |A| \cdot |C|$.

Proof: We know that α is injective and β is surjective and moreover $\mathrm{im}(\alpha) = \ker(\beta)$.
For any $c \in C$ the fibre $\beta^{-1}(c)$ contains exactly $|A|$ elements. Because: Since β is surjective there exists $b \in \beta^{-1}(c)$. And since $\mathrm{im}(\alpha) = \ker(\beta)$ it follows that $b + \mathrm{im}(\alpha) \in \beta^{-1}(c)$. As α is injective this implies that $|\beta^{-1}(c)| \geq |A|$. Assume $x \in B \setminus b + \mathrm{im}(\alpha)$ is another element with $\beta(x) = c$ then $x - b \in \ker(\beta) = \mathrm{im}(\alpha)$ and thus $x \in b + \mathrm{im}(\alpha)$. Contradiction.
So $|\beta^{-1}(c)| = |A|$. Since $B = \bigcup_{c \in C} \beta^{-1}(c)$ it follows that $|B| = |C| \cdot |A|$. □