Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

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11. November 2020

Introduction to Commutative Algebra and Algebraic Geometry
Exercise Sheet 2

Exercise 1.
Let K be a field. Consider $R := K[T^2, T^3] \subset K[T]$ and prove the following statements:

1. R is noetherian.
2. $\sqrt{T^2} = \langle T^2, T^3 \rangle$.

Exercise 2.
Let (X, Ω) be a topological space. The closure of a subset $A \subset X$ is the intersection $\overline{A}$ of all closed subsets $B \subset X$ with $A \subset B$. Show:

1. A subset $A \subset X$ is closed in X if and only if $A = \overline{A}$.
2. For every finite union $A := A_1 \cup \ldots \cup A_n$ of subsets $A_1, \ldots, A_n \subset X$ the following applies: $\overline{A} = \overline{A_1} \cup \ldots \cup \overline{A_n}$.
3. Let $A \subset B \subset X$ be subsets. The closure of A in B with respect to the subspace topology is given by $\overline{A} \cap B$. The *subspace topology* of $B \subset X$ is given by the following system of open sets: $\Omega_B = \{Y \cap B | Y \subset X \text{open}\}$.

Exercise 3.
1. Let $V_1 = \{1, 3, 5\} \subset \mathbb{R}$ and $V_2 = \{1, 2, 3, 4, 5\} \subset \mathbb{R}$. Compute $I(V_1) \subset \mathbb{R}[x]$ and $I(V_2) \subset \mathbb{R}[x]$. Prove $I(V_2) \subset I(V_1)$.
2. Let $V_3 = \{(0, 0), (0, 2), (1, 0), (1, 1)\}, V_4 = \{(1, -1), (1, 0), (0, 0), (3, 1), (0, 2)\} \subset \mathbb{R}^2$. Compute $I(V_3), I(V_4) \subset \mathbb{R}[x, y]$ and $I(V_3 \cap V_4)$.
3. Let $M \neq \emptyset$ be an arbitrary finite subset of $\mathbb{R}^2$. Find a polynomial $f \in \mathbb{R}[x, y]$ such that $M = V(f)$.
4. Consider $\mathbb{Z} \subset \mathbb{R}$. Compute $I(\mathbb{Z}) \subset \mathbb{R}[x]$.

Exercise 4.
Let A, B, C be finite groups and let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be a short exact sequence of group homomorphisms, i.e., α is injective, β is surjective and $\ker(\beta) = \operatorname{im}(\alpha)$. Prove $|B| = |A| \cdot |C|$.

Submission: In groups of up to three students, until Wednesday, 18. November 2020, 12:00 o'clock via URM. You are allowed to submit your solutions in German.