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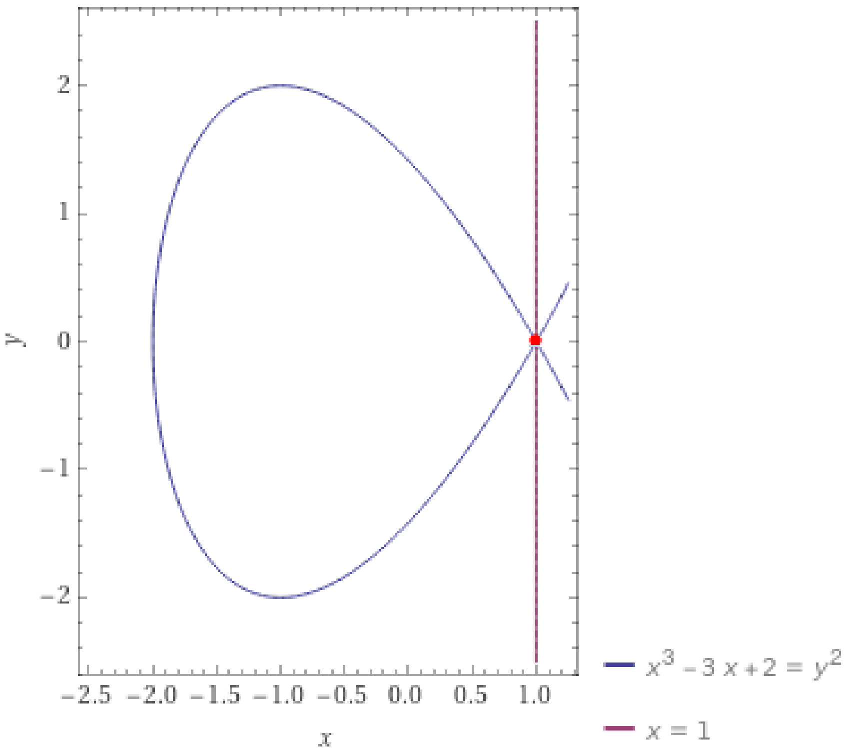
Introduction to Commutative Algebra and Algebraic Geometry
Exercise Sheet 3 - Solution

Exercise 1.
Let $X_1 = V(x^3 - 3x + 2 - y^2)$, $X_2 = V(x-1)$ be two varieties in $\mathbb{C}^2$. Make a sketch of the real part of $X_1, X_2 \subset \mathbb{R}^2$ in the same coordinate system and compute $\sqrt{I(X_1) + I(X_2)} \subset \mathbb{C}[X, Y]$.

Solution: We know: $I(X_1) = \sqrt{\langle x^3 - 3x + 2 - y^2 \rangle} = \langle x^3 - 3x + 2 - y^2 \rangle$ and $I(X_2) = \sqrt{\langle x - 1 \rangle} = \langle x - 1 \rangle$,



because both polynomials are irreducible.

$$\begin{aligned} I(X_1) + I(X_2) &= \langle x^3 - 3x + 2 - y^2 \rangle + \langle x - 1 \rangle \\ &= \langle x^3 - 3x + 2 - y^2, x - 1 \rangle \\ &= \langle (x^2 + x - 2)(x - 1) - y^2, x - 1 \rangle \\ &= \langle y^2, x - 1 \rangle \end{aligned}$$

It follows:

$$\sqrt{I(X_1) + I(X_2)} = \langle y, x - 1 \rangle$$

Looking at the sketch we see that $X_1 \cap X_2 = \{(1, 0)\}$ so $I(X_1 \cap X_2) = \langle y, x - 1 \rangle$ which fits with Prop. 1.4.14.

Exercise 2.
Let K be a field. Consider the variety $V = V(Y^2 - X^3)$ of the cuspidal cubic $Y^2 - X^3 \in K[X, Y]$. Let $K[V]$ be the coordinate ring of V. Prove

$$K[V] \cong K[T^2, T^3].$$

Proof: By definition $K[V] = K[X, Y]/I(V)$. For K is algebraically closed we know $I(V) = \sqrt{\langle Y^2 - X^3 \rangle}$ and since $Y^2 - X^3$ is irreducible (by Ex. 3) $I(V) = \langle Y^2 - X^3 \rangle$.

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So we have $K[V] = K[X,Y]/\langle Y^2 - X^3\rangle$. We consider the following K-algebra homomorphism:

$$\phi : K[X,Y] \to K[T]$$
$$X \mapsto T^2$$
$$Y \mapsto T^3$$

We compute image and kernel of ϕ:

$$\text{im}(\phi) = \{p \in K[T] | p \text{ only contains exponents of } T \text{ divisble by 2 or 3}\} = K[T^2, T^3]$$
$$\ker(\phi) = \{p(X,Y) \in K[X,Y] | \phi(p)(T) = p(T^2, T^3) = 0\}$$

To compute the generators of the ideal $\ker(\phi)$ we use polynomial division in Y with $Y^2 - X^3$ over $K[X][Y]$: We can write any $p \in K[X,Y]$ as

$$p = g \cdot (Y^2 - X^3) + Y \cdot r_1 + r_2$$

where $g \in K[X,Y]$ and $r_1, r_2 \in K[X]$.
In particular we can write $p \in \ker(\phi)$ like this. It follows that:

$$\begin{aligned} 0 = \phi(p) &= \phi(g(X,Y) \cdot (Y^2 - X^3) + Y \cdot r_1(X) + r_2(X)) \\ &= g(T^2, T^3) \cdot ((T^3)^2 - (T^2)^3) + T^3 \cdot r_1(T^2) + r_2(T^2) \\ &= T^3 \cdot r_1(T^2) + r_2(T^2) \end{aligned}$$

Now $T^3 \cdot r_1(T^2)$ contains only odd exponents of T while $r_2(T^2)$ contains only terms with even exponents of T. Therefore $T^3 \cdot r_1(T^2) + r_2(T^2) = 0$ implies $r_1 = r_2 = 0$ and $\ker(\phi) = \langle Y^2 - X^3\rangle$.
It follows:

$$K[V] = K[X,Y]/\langle X^3 - Y^2\rangle = K[X,Y]/\ker(\phi) \cong \text{im}(\phi) = K[T^2, T^3]$$

□

Exercise 3.
Let K be an algebraically closed field. Find all polynomials $f \in K[X_1, \dots, X_n]$ for which $V(X_{n+1}^2 - f) \subset K^{n+1}$ is irreducible. Prove your claim.

Proof: For char$(K) \neq 2$ we claim that $V(X_{n+1}^2 - f)$ is irreducible if and only if $f \in K[X_1, \dots, X_n] \setminus \{0\}$ is not a square or $f = 0$.
If $f = 0$, then $V(X_{n+1}^2) = V(X_{n+1})$ is irreducible. So lets assume $f \neq 0$.
Let $g \in K[X_1, ..., X_n]$ be a polynomial, $g \neq 0$. Choose $f = g^2$. Then $X_{n+1}^2 - f = X_{n+1}^2 - g^2 = (X_{n+1} - g)(X_{n+1} + g)$ is reducible with two different factors (since char$(K) \neq 2$). So $V(X_{n+1}^2 - f) = V(X_{n+1} - g) \cup V(X_{n+1} + g)$ is reducible.
Now let $V(X_{n+1}^2 - f)$ be reducible. (This implies $f \neq 0$ since for $f = 0$ the variety is irreducible.) Since this variety is a hypersurface this implies that $X_{n+1}^2 - f$ is reducible with two different factors, see Example 1.5.9. So we can write $X_{n+1}^2 - f = gh$ with $g \neq h \in K[X_,1..., X_{n+1}]$ non-constant.
The degree of gh in X_{n+1} is 2. If the degree of g in X_{n+1} were 2, then f would have to contain terms with X_{n+1}. Contradiction to $f \in K[X_1, ..., X_n]$.
So it follows that the degree of g and h in X_{n+1} has to be one: $\deg_{x_{n+1}}(g) = \deg_{x_{n+1}}(h) = 1$. So we can write $g = p_1 X_{n+1} + q_1$, $h = p_2 X_{n+1} + q_2$ with $p_1, p_2, q_1, q_2 \in K[X_1, ..., X_n]$. Then:

$$\begin{aligned} X_{n+1}^2 - f = gh &= (p_1 X_{n+1} + q_1)(p_2 X_{n+1} + q_2) \\ &= p_1 p_2 X_{n+1}^2 + (p_1 q_2 + p_2 q_1) X_{n+1} + q_2 q_1 \end{aligned}$$

Comparing the right and left side of the equation above we see: $p_1 p_2 = 1$ and $p_1 q_2 + p_2 q_1 = 0$. It follows that $p_1, p_2 \in K \setminus \{0\}$ and $q_2 = -\frac{p_2}{p_1} q_1$.
So: $f = q_2 q_1 = -\frac{p_2}{p_1} q_1^2$. Since K is algebraically closed there exits $a \in K$ such that $a^2 = -\frac{p_2}{p_1}$. Therefore $f = (a q_1)^2$.

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Hence, For char$(K) \neq 2$ we have $X_{n+1}^2 - f$ with $f \in K[X_1, ..., x_n]$ is irreducible if and only if f is not a square of a polynomial in $K[X_1, ..., X_n] \setminus \{0\}$ or $f = 0$.

Looking at the above proof we see that the implication: $X_{n+1}^2 - f$ reducible $\Rightarrow f$ is a square works exactly the same if char$(K) = 2$. Only now the two factors we obtain are the same because of $(X_{n+1} - g) = (X_{n+1} + g)$ in char$(K) = 2$.
We still know $V(X_{n+1}^2 - f)$ is reducible if only if $X_{n+1}^2 - f$ is reducible with different factors.
Considering the above, we know that $X_{n+1}^2 - f$ reducible implies that f is a square, which in this case means, that $X_{n+1}^2 - f$ is a square. Hence, $X_{n+1}^2 - f$ is never product of two different factors in char$(K) = 2$, so for all $f \in K[X_1, \ldots, X_n]$ the variety $V(X_{n+1}^2 - f)$ is irreducible. □

Exercise 4.
Let K be a field and let $0 \to V_0 \to \ldots \to V_n \to 0$ be an exact sequence of finite dimensional vector spaces over K. Prove:

$$\sum_{i=0}^{n} (-1)^i \dim(V_i) = 0.$$

Proof: We write

$$0 \xrightarrow{\phi_{-1}} V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_{n-1}} V_n \to 0.$$

We know that

$$0 \to \ker(\phi_i) \hookrightarrow V_i \xrightarrow{\phi_i} \mathsf{im}(\phi_i) \to 0$$

is a short exact sequence. From linear algebra we know that for a K-linear map $\phi_i : V_i \to \mathsf{im}(\phi_i)$ the following holds: $\dim(V_i) = \dim(\ker(\phi_i)) + \dim(\mathsf{im}(\phi_i))$. By the exactness of the long sequence we know:

$$\dim(V_i) = \dim(\mathsf{im}(\phi_{i-1})) + \dim(\mathsf{im}(\phi_i)).$$

Now we can compute the sum:

$$\begin{aligned}
\sum_{i=0}^{n-1} (-1)^i \dim(V_i) &= \sum_{i=0}^{n-1} (-1)^i (\dim(\ker(\phi_i)) + \dim(\mathsf{im}(\phi_i))) \\
&= \sum_{i=0}^{n-1} (-1)^i (\dim(\mathsf{im}(\phi_{i-1})) + \dim(\mathsf{im}(\phi_i))) \\
&= \dim(\mathsf{im}(\phi_{-1})) + \sum_{i=1}^{n-1} (-1)^i \dim(\mathsf{im}(\phi_{i-1})) + \sum_{i=0}^{n-1} (-1)^i (\dim(\mathsf{im}(\phi_i))) \\
{\scriptstyle \dim(\mathsf{im}(\phi_{-1}))\,=\,0} &= \sum_{i=1}^{n-1} (-1)^i \dim(\mathsf{im}(\phi_{i-1})) + \sum_{i=0}^{n-2} (-1)^i (\dim(\mathsf{im}(\phi_i))) + (-1)^{n-1} \dim(\mathsf{im}(\phi_{n-1}) \\
{\scriptstyle \textbf{index shift}} &= \sum_{i=0}^{n-2} (-1)^{i+1} \dim(\mathsf{im}(\phi_i)) + \sum_{i=0}^{n-2} (-1)^i (\dim(\mathsf{im}(\phi_i))) + (-1)^{n-1} \dim(\mathsf{im}(\phi_{n-1}) \\
&= (-1)^{n-1} \dim(\mathsf{im}(\phi_{n-1}) \\
{\scriptstyle \mathsf{im}(\phi_{n-1})\,=\,V_n} &= -(-1)^n \dim(V_n) \\
&\Rightarrow \sum_{i=0}^{n} (-1)^i \dim(V_i) = 0
\end{aligned}$$

□