Eberhard-Karls-Universität Tübingen
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9. December 2020

Introduction to Commutative Algebra and Algebraic Geometry
Exercise Sheet 6

Exercise 1 (*Product criterion*).
Let K be a field, $>$ be a monomial order, $f, g \in K[\underline{x}]$, $\gcd(\mathrm{LM}(f), \mathrm{LM}(g)) = 1$.
Show that there is a polynomials division with remainder of $\mathrm{spoly}(f, g)$ by (f, g) with remainder 0.

Hint: Show first that spoly$(f, g) = a_0 f + b_0 g$ for $a_0 = -tail(g)$ and $b_0 = tail(f)$ and then define recursively $a_i = tail(a_{i-1})$ and $b_i = tail(b_{i-1})$. Consider the maximal value N such that $u \cdot spoly(f, g) = a_N f + b_N g$ for some element $u \in K[\underline{x}]^*$ and distinguish the two cases that LT$(a_N f)$ + LT$(b_N g)$ vanishes respectively does not vanish.

Exercise 2.
The degree lexicographical ordering $>_{Dp}$ on Mon_n is defined by

$$\underline{x}^\alpha >_{Dp} \underline{x}^\beta :\Leftrightarrow |\alpha| > |\beta| \text{ or } (|\alpha| = |\beta| \text{ and } \exists k : \alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k).$$

A polynomial $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \underline{x}^\alpha \in K[x_1, \ldots, x_n]$ is called *homogeneous* if for all α with $a_\alpha \neq 0$ the absolute value $|\alpha|$ is constant.
Show that a monomial ordering $>$ on Mon_n equals $>_{Dp}$ if and only if $>$ is a degree ordering and for any homogeneous $f \in K[\underline{x}]$ with $\mathrm{LM}(f) \in K[x_k, \ldots, x_n]$ we have $f \in K[x_k, \ldots x_n], k = 1, \ldots, n$.

Exercise 3.
Apply IDBuchberger to the following triple $(g, G, >)$:

$$g = x^4 + y^4 + z^4 + xyz,\ G = \{\partial g/\partial x, \partial g/\partial y, \partial g/\partial z\},\ >_{dp} .$$

Submission: In groups of up to three students, until Wednesday, 16. December 2020, 12:00 o'clock via URM. You are allowed to submit your solutions in German.