Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

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13. January 2021

Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 8

Exercise 1.
Let $f, g \in K[x]$, the polynomial ring in *one* variable. Express the greatest common divisor and the least common multiple of f and g in terms of elements in $\mathsf{syz}(f, g)$ and derive an algorithm to compute these, assuming we can compute a Gröbner basis of $\mathsf{syz}(f, g)$.

Solution: Let $d = \mathsf{gcd}(f, g)$, then there exist $k, l \in K[x]$ such that $f = kd$, $g = ld$. The element $\begin{pmatrix} -l \\ k \end{pmatrix}$ is then an element in $\mathsf{syz}(f, g)$ which determines the gcd of f and g uniquely (up to units).
Let $c = \mathsf{lcm}(f, g)$ then there exist $a, b \in K[x]$ such that $af = c$, $bg = c$ and the element $\begin{pmatrix} a \\ -b \end{pmatrix}$ is in $\mathsf{syz}(f, g)$ and determines the lcm uniquely (up to units).
We show first, that for any $f, g \in K[x]$ the syzygy module $\mathsf{syz}(f, g)$ is a rank one module, i.e., generated by one element.
Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathsf{syz}(f, g)$. Then

$$\alpha f = -\beta g$$
$$\Rightarrow \alpha \frac{f}{\mathsf{gcd}(f, g)} = -\beta \frac{g}{\mathsf{gcd}(f, g)}$$

Since $\frac{f}{\mathsf{gcd}(f,g)}$ and $\frac{g}{\mathsf{gcd}(f,g)}$ are coprime, we see that α is a multiple of $\frac{g}{\mathsf{gcd}(f,g)}$ and β is a multiple of $\frac{f}{\mathsf{gcd}(f,g)}$.
Since $K[x]$ is a factorial ring, we can use primefactorization. Together with $\alpha f = -\beta g$ it follows that the factor for both α and β is the same.

$$\alpha = d \cdot \frac{g}{\mathsf{gcd}(f, g)}, \ \beta = -d \cdot \frac{f}{\mathsf{gcd}(f, g)}.$$

As $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ was a random element in $\mathsf{syz}(f, g)$ it follows that $\mathsf{syz}(f, g) = \langle \begin{pmatrix} \frac{g}{\mathsf{gcd}(f,g)} \\ -\frac{f}{\mathsf{gcd}(f,g)} \end{pmatrix} \rangle$.
Furthermore, this generating set is a reduced Gröbner basis of $\mathsf{syz}(f, g)$ (since it consists of only one element). Thus, it is unique up to units.

Algorithm:
INPUT: $f, g \in K[\underline{x}]$
OUPUT: $(\mathsf{lcm}(f, g), \mathsf{gcd}(f, g))$
INSTRUCTIONS:

- Compute a reduced Gröbner basis $S = (\begin{pmatrix} a \\ b \end{pmatrix})$ of $\mathsf{syz}(f, g)$.
- $\mathsf{lcm}(f, g) = af$, $\mathsf{gcd}(f, g) = \frac{fg}{\mathsf{lcm}(f,g)}$

Return $(\mathsf{lcm}(f, g), \mathsf{gcd}(f, g))$. □

Exercise 2.
Let

$$V = V((x + y) \cdot (x - y) \cdot (x + z^2)) \subset \mathbb{C}^3$$

and

$$W = V((x + z^2) \cdot (x - y) \cdot (z + y)) \subset \mathbb{C}^3.$$

Draw sketches of V and W in $\mathbb{R}^3$, and try to guess the ideal of $\overline{V \setminus W}$. How could you verify your guess? Answer with an approriate algorithm and describe the steps accurately without doing all the computations.

Proof: We rewrite V and W:

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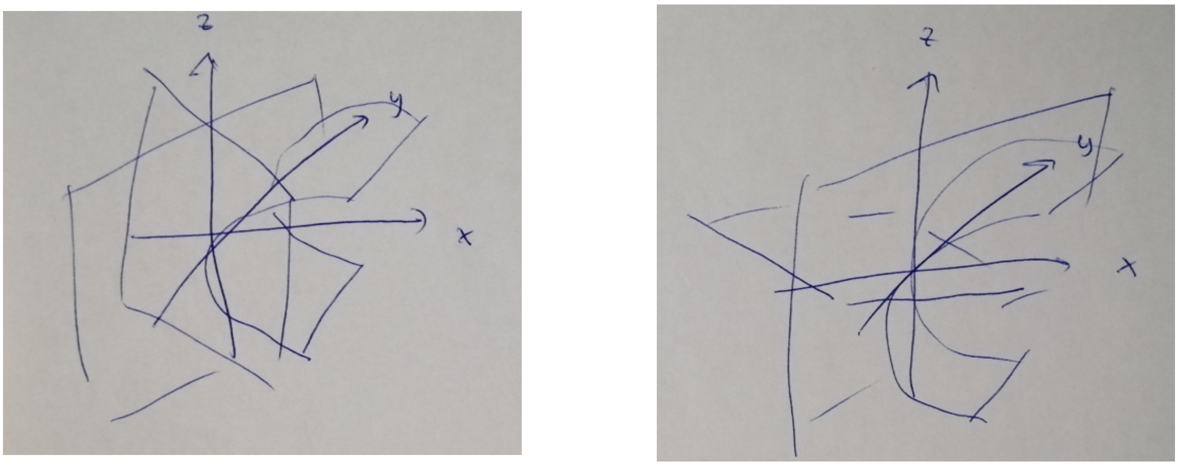

(a) $V \cap \mathbb{R}^3$ (b) $W \cap \mathbb{R}^3$

Abbildung 1

$$\begin{aligned} V &= V((x+y)\cdot(x-y)\cdot(x+z^2)) \\ &= V(x+y) \cup V(x-y) \cup V(x+z^2) \\ W &= V((x+z^2)\cdot(x-y)\cdot(z+y)) \\ &= V(x+z^2) \cup V(x-y) \cup V(z+x) \end{aligned}$$

For the sketch see Figure 1.
The variety $V(x+y)$ does not seem to be contained in W so we guess $\overline{V \setminus W} = V(x+y)$.
We need to verify this claim by checking $I = \sqrt{I}$ (1.5.13) and computing the ideal quotient $(I : J)$ of

$$I = \langle (x+y)\cdot(x-y)\cdot(x+z^2)\rangle = \langle x^3 + x^2z^2 - xy^2 - y^2z^2\rangle$$

and

$$J = \langle (x+z^2)\cdot(x-y)\cdot(z+y)\rangle = \langle x^2z +^2 y - xyz - xy^2 + z^3 + yz^2 - yz^3 - y^2z^2\rangle.$$

Since I is a principal ideal, we know that for some $r \in \sqrt{I}$ here is an $n \in \mathbb{N}, f \in K[x,y,z]$ such that $r^n = f(x+y)(x-y)(x+z^2)$. Since they are prime elements $(x+y)$, $(x-y)$, $(x+z^2)$ all have to divide r, so $r \in I$ and $I = \sqrt{I}$.
Computing the ideal quotient $(I : J)$ using Variant 1 of algorithm 2.5.9, we first compute a Gröbner basis for $I \cap J$ with algorithm 2.5.7. For that we have to eliminate t from

$$G = \langle t \cdot (x^3 + x^2z^2 - xy^2 - y^2z^2), (1-t)(x^2z + x^2y - xyz - xy^2 + z^3 + yz^2 - yz^3 - y^2z^2)\rangle.$$

We choose $>_{lp}$ with $t > x > y$ as our global elimination order and compute a Gröbner basis B of G.
We write $B = (g_1, ..., g_k)$ for some k and $g = x^2z +^2 y - xyz - xy^2 + z^3 + yz^2 - yz^3 - y^2z^2$ (then $J = \langle g\rangle$.) We then set $g_i' = \frac{g_i}{g}$ and $S = (g_1', \ldots g_k')$. This is a generating set of $I : J$ by algorithm 2.5.9. □

Exercise 3. a) Is the polynomial $f = x^7 + x^2$ in the radical of $I = \langle x^{11} + x^6, yx^4 + x^8, y^3 + x^2\rangle$? Use algorithm 2.5.5 to prove your claim.

Hint: Eliminate y first.

Solution: Since $f \in K[x]$, we know that $f \in \sqrt{I}$ if and only if $f \in \sqrt{I \cap K[x]}$.

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We choose the global elimination monomial ordering $>_{lp}$ with $y > x$ and compute a grobner basis for I.

$$\begin{aligned}
f_1 &= x^{11} + x^6 \\
f_2 &= yx^4 + x^8 \\
f_3 &= y^3 + x^2 \\
P &= \{(f_1, f_2), (f_1, f_3), (f_2, f_3)\} \\
\text{Using product criterion we eliminate } (f_1, f_3) & \\
\mathsf{spoly}(f_1, f_2) &= y(x^{11} + x^6) - x^7(yx^4 + x^8) \\
&= -x^{15} + yx^6 \\
\mathsf{LM}(f_2) | \mathsf{LM}(-x^{15} + yx^6) & \\
q_2 &= x^2 \\
r &= yx^6 - x^{15} - x^2(yx^4 + x^8) \\
&= -x^{15} - x^{10} = -x^4(x^{11} + x^6) \\
\Rightarrow P &= \{(f_2, f_3)\} \\
\mathsf{spoly}(f_2, f_3) &= y^2(yx^4 + x^8) - x^4(y^3 + x^2) \\
&= y^2x^8 - x^6 \\
\mathsf{LM}(f_2) | \mathsf{LM}(y^2x^8 - x^6) & \\
q_2 &= yx^4 \\
r &= y^2x^8 - x^6 - yx^4(yx^4 + x^8) \\
&= -x^6 - yx^{12} \\
q_2 &= yx^4 - x^8 \\
r &= -x^6 - yx^{12} + x^8(yx^4 + x^8) \\
&= -x^{16} + x^6 \\
q_1 &= -x^5 \\
r &= -x^{16} + x^6 + x^5(x^{11} + x^6) \\
&= x^6 + x^{11} = f_1 \\
\Rightarrow P &= \emptyset
\end{aligned}$$

So I is already given by a Gröbner basis. Hence $I \cap K[x] = \langle x^{11} + x^6 \rangle$.
Now we apply Algorithm 2.5.5 to $f = x^7 + x^2$ and $J = \langle x^{11} + x^6 \rangle$.
We choose $>_{lp}$ with $t > x$ as the global elimination order. We need to compute a minimal Gröbner basis for

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$\langle x^{11} + x^6, 1 - t(x^7 + x^2)\rangle$.

$$f_1 = x^{11} + x^6$$
$$f_2 = 1 - t(x^7 + x^2)$$
$$S = \{f_1, f_2\}$$
$$P = \{(f_1, f_2)\}$$
$$\mathsf{spoly}(f_1, f_2) = -t(x^{11} + x^6) - x^4(1 - t(x^7 + x^2)) = -x^4$$

this cannot be divided by any Leading monomial in S so we add it to it.

$$f_3 = -x^4$$
$$S = \{f_1, f_2, f_3\}$$
$$P = \{(f_1, f_3), (f_2, f_3)\}$$
$$\mathsf{spoly}(f_1, f_3) = -(x^{11} + x^6) + x^7(x^4) = -x^6$$

$\mathsf{LM}(f_3)|x^6 \Rightarrow q_3 = x^2$

$r = -x^6 - x^2x^4 = 0$

$$\mathsf{spoly}(f_2, f_3) = (-t(x^7 + x^2) + 1) + tx^3(x^4) = -tx^2 + 1$$

this cannot be divided by any Leading monomial in S so we add it to it.

$$f_4 = -tx^2 + 1$$
$$S = \{f_1, f_2, f_3, f_4\}$$
$$P = \{(f_1, f_4), (f_2, f_4), (f_3, f_4)\}$$
$$\mathsf{spoly}(f_1, f_4) = t(x^{11} + x^6) + x^9(-tx^2 + 1) = tx^6 + x^9$$

$\mathsf{LM}(f_3)|tx^6$

$$r = tx^6 + x^9 - tx^2x^4 = x^9 = x^5(x^4).$$
$$\mathsf{spoly}(f_2, f_4) = -(-t(x^7 + x^2) + 1) + x^5(-tx^2 + 1) = tx^2 - 1 + x^5$$

$\mathsf{LM}(f_4)|tx^2$

$$r = tx^2 - 1 + x^5 - tx^2 + 1 = x^5 = x(x^4).$$
$$\mathsf{spoly}(f_3, f_4) = -t(-x^4) + x^2(-tx^2 + 1) = x^2$$

this cannot be divided by any Leading monomial in S so we add it to it.

$$f_5 = x^2$$
$$S = \{f_1, f_2, f_3, f_4, f_5\}$$
$$P = \{(f_1, f_5), (f_2, f_5), (f_3, f_5), (f_4, f_5)\}$$
$$\mathsf{spoly}(f_1, f_5) = x^{11} + x^6 - x^9(x^2) = x^6 = x^2x^4$$
$$\mathsf{spoly}(f_2, f_5) = -t(x^7 + x^2) + 1 + tx^5(x^2) = -tx^2 + 1 = f_4$$
$$\mathsf{spoly}(f_3, f_5) = -x^4 + x^2(x^2) = 0$$
$$\mathsf{spoly}(f_4, f_5) = -tx^2 + 1 + t(x^2) = 1$$

$\Rightarrow$ So we have to add 1 to S.

$P = \emptyset$.(here we use that by the product criterion with 1 there is always polynomial division with remainder 0

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So $S = \{f_1, f_2, f_3, f_4, f_5, 1\}$ is a Gröbner basis for $\langle x^{11} + x^6, 1 - t(x^7 + x^2)\rangle$. For the algorithm we need a minimal Gröbner basis, but we can already see that 1 will still be contained in the reduced Gröbner basis, so the output of the algorithm will be 1, i.e., $f \in \sqrt{I}$. □

b) Let $I = \langle x^2 + 2y^2 - 3, x^2 + xy + y^2 - 3\rangle \subset \mathbb{Q}[x, y]$. Compute $I \cap \mathbb{Q}[y]$.

Solution: We choose the gloabl elimination ordering $>_{lp}$ with $x > y$. We compute a Gröbner basis of I.

$$
\begin{aligned}
f_1 &= x^2 + 2y^2 - 3\\
f_2 &= x^2 + xy + y^2 - 3\\
S &= \{f_1, f_2\}\\
P &= \{(f_1, f_2)\}\\
\text{spoly}(f_1, f_2) &= f_1 - f_2 = x^2 + 2y^2 - 3 - (x^2 + xy + y^2 - 3)\\
&= y^2 - xy
\end{aligned}
$$

this cannot be divided by any Leading monomial in S so we add it to it.

$$
\begin{aligned}
f_3 &= -xy + y^2\\
S &= \{f_1, f_2, f_3\}\\
P &= \{(f_1, f_3), (f_2, f_3)\}\\
\text{spoly}(f_1, f_3) &= -yf_1 - xf_3 = -yx^2 - 2y^3 + 3y - x(-xy + y^2)\\
&= -2y^3 + 3y - xy^2\\
q_3 &= y\\
r &= -xy^2 - 2y^3 + 3y - y(-xy + y^2)\\
&= -2y^3 + 3y - y(y^2)\\
&= -3y^3 + 3y
\end{aligned}
$$

this cannot be divided by any Leading monomial in S so we add it to it.

$$
\begin{aligned}
f_4 &= -3y^3 + 3y\\
S &= \{f_1, f_2, f_3, f_4\}\\
P &= \{(f_2, f_3), (f_1, f_4), (f_2, f_4), (f_3, f_4)\}
\end{aligned}
$$

use Product criterion to delete $(f_1, f_4), (f_2, f_4)$

$$
\begin{aligned}
P &= \{(f_2, f_3), (f_3, f_4)\}\\
\text{spoly}(f_2, f_3) &= -yf_2 - xf_3 = -y(x^2 + xy + y^2 - 3) - x(-xy + y^2)\\
&= -2xy^2 - y^3 + 3y\\
q_3 &= 2y\\
r &= -2xy^2 - y^3 + 3y - 2y(-xy + y^2)\\
&= -y^3 + 3y - 2y^3 = -3y^3 + 3y = f_4\\
\text{spoly}(f_3, f_4) &= -3y^2 f_3 + xf_4 = -3y^2(-xy + y^2) + x(-3y^3 + 3y)\\
&= -3y^4 + 3xy\\
q_3 &= -3\\
r &= -3y^4 + 3xy + 3(-xy + y^2)\\
&= -3y^4 + 3y^2 = y(-3y^3 + 3y) = yf_4
\end{aligned}
$$

$\Rightarrow P = \emptyset$.

So a Gröbner basis of I is given by $S = \{x^2 + 2y^2 - 3, x^2 + xy + y^2 - 3, -xy + y^2, -3y^3 + 3y, \}$. The only element of this Gröbner basis that has leading monomial in $\mathbb{Q}[y]$ is $-3y^3 + 3y$. So $I \cap \mathbb{Q}[y] = \langle -3y^3 + 3y\rangle$. □