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Wintersemester 2020/2021
20. January 2021

Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 9

Exercise 1.
Let K be an algebraically closed field, $A := K[T_1, T_2]/\langle T_1 \cdot T_2\rangle$ and $f \in A$ the equivalence class of T_1. Prove: $A_f \cong K[T]_T$.

Proof: First we prove that for a ring R with ideal $I \subset R$ and multiplicatively closed set $S \subset R$ the following holds:

$$\overline{S}^{-1}(R/I) = S^{-1}R/S^{-1}I$$

where $\overline{S}$ is the set of residue classes of elements of S modulo I. If $S \cap I \neq \emptyset$ we have $S^{-1}R = 0$ and then $\overline{S} = 0$ as well, which means $\overline{S}^{-1}(R/I)$, so we can write S instead of $\overline{S}$ in the following.
Consider the exact sequence $0 \to I \to R \to R/I \to 0$. If the corresponding localised sequence $0 \to S^{-1}I \to S^{-1}R \to S^{-1}(R/I) \to 0$ is also exact, the claim follows from the Homomorphism Theorem. So we need to prove that $\text{im}(S^{-1}I \to S^{-1}R) = \ker(S^{-1}R \to S^{-1}(R/I))$ (exactness at the other two steps follows directly from the definition of the maps). Denote $\varphi : I \to R$, $\psi : R \to R/I$, then $S^{-1}I \to S^{-1}R$ is given by $S^{-1}\varphi$ and $S^{-1}R \to S^{-1}(R/I)$ is given by $S^{-1}\psi$, where $S^{-1}\varphi(\frac{r}{s}) = \frac{\varphi(r)}{s}$ and analogously for $S^{-1}\psi$. These are again ringhomorphisms.

$$S^{-1}\varphi : S^{-1}I \to S^{-1}R$$
$$S^{-1}\psi : S^{-1}R \to S^{-1}(R/I)$$

$\text{im}(S^{-1}\varphi) \subset \ker(S^{-1}\psi)$:

$$S^{-1}\psi \circ S^{-1}\varphi = S^{-1}(\psi \circ \varphi) = 0_{\text{map}}$$

$\text{im}(S^{-1}\varphi) \supset \ker(S^{-1}\psi)$: Let $\frac{r}{s} \in \ker(S^{-1}\psi)$. Then $\frac{0}{1} = S^{-1}\psi(\frac{r}{s}) = \frac{\psi(r)}{s}$, which is an identity in $S^{-1}(R/I)$.
$\Rightarrow \exists \overline{u} \in \overline{S}$: $\overline{u} \cdot \overline{1} \cdot \psi(r) = \overline{u} \cdot \overline{s} \cdot \overline{0} = \overline{0}$. Furthermore $\exists u \in S$: $\psi(u) = \overline{u}$
$\Rightarrow u \cdot r \in \ker(\psi) = \text{im}(\varphi)$
$\Rightarrow \exists m \in I$: $\varphi(m) = u \cdot r$
$\Rightarrow \frac{r}{s} = \frac{ur}{us} = \frac{\varphi(m)}{us} = S^{-1}\varphi(\frac{m}{us}) \in \text{im}(S^{-1}\varphi)$.
Applying this to the situation in the exercise we set $S = \{T_1^n | n \in \mathbb{N}\}$ and obtain: $A_f \cong S^{-1}K[T_1, T_2]/S^{-1}\langle T_1 \cdot T_2\rangle$.
Now we can compute

$$\begin{aligned} S^{-1}\langle T_1 \cdot T_2\rangle &= \{\frac{g \cdot T_1 \cdot T_2}{T_1^n} | g \in K[T_1, T_2], n \in \mathbb{N}\} \\ &= \langle T_2\rangle_{T_1} \\ &= S^{-1}\langle T_2\rangle \end{aligned}$$

Since $S^{-1}K[T_1, T_2] = K[T_1, T_2]_{T_1}$ we obtain the following isomorphism:

$$A_f \cong S^{-1}K[T_1, T_2]/S^{-1}\langle T_1 \cdot T_2\rangle = K[T_1, T_2]_{T_1}/\langle T_2\rangle_{T_1} \cong K[T_1]_{T_1} \cong K[T]_T.$$

□

Exercise 2.
Let R be a commutative ring with one and let $S \subset R$ be multiplicatively closed. Show: If R is noetherian, then $S^{-1}R$ is noetherian.

Proof: Let R be a noetherian commutative ring with one and let $S \subset R$ be multiplicatively closed.
We prove the following lemma: If J is an ideal in $S^{-1}R$, then $J = S^{-1}(J \cap R)$.
Let $a \in S^{-1}(J \cap R)$. So $a = \frac{r}{s}$ with $r \in R \cap J$, $s \in S$. In particular, $a \in J \subset S^{-1}R$.
Let $\frac{r}{s} \in J \subset S^{-1}R$. Then $\frac{r}{1} = \frac{s}{1} \cdot \frac{r}{s} \in J$ so $r \in J \cap R$. Thus $\frac{r}{s} \in S^{-1}(J \cap R)$.

Now let

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots$$

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be an ascending chain of ideals in $S^{-1}R$. Then $I_n = J_n \cap R$ is an ideal in R for all $n \in \mathbb{N}$. The inclusions are maintained so

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

is an ascending chain of ideals in R.
$\Rightarrow \exists k \in \mathbb{N} : I_k = I_m \, \forall m \geq k$
Since $S^{-1}(J_n \cap R) = J_n$ by the lemma we proved in the beginning, it follows that $J_k = S^{-1}I_k = S^{-1}I_m = J_m$ for all $m \geq k$. So $S^{-1}R$ is noetherian. □

Exercise 3.
Let K be an algebraically closed field. Determine the algebra of regular functions $\mathcal{O}(U_i)$ for the following open sets U_i:

- $U_1 = K \setminus \{0\} \subset K$,
- $U_2 = K^\times \times K \subset K^2$,
- $U_3 = K^\times \times K^\times \subset K^2$.

Solution: First we note, that $K[K](= K[\mathbb{A}_K]) = K[x]$ and that $K[K^2](= K[\mathbb{A}^2_K]) = K[x_1, x_2]$.

- $U_1 = K \setminus \{0\} \subset K$:
  By Definition 3.2.3 we know that $\mathcal{O}(U_1) = \{f : U_1 \to K \text{ regular }\}$. We can write $K \setminus \{0\} = K \setminus V(x) = D(x)$. Using 3.2.9 we obtain $\mathcal{O}(U_1) = \mathcal{O}(K \setminus \{0\}) = K[x]_x$.
- We use that $U_2 = K^\times \times K = K^2 \setminus \{(0,a) | a \in K\} = K^2 \setminus V(x_1)$ when we associate x_1 with the first cooordinate and x_2 with the second. Again using 3.2.9 we obtain $\mathcal{O}(U_2) = K[x_1, x_2]_{x_1}$.
- We use that $U_3 = K^\times \times K^\times = K^2 \setminus (\{(0,a)|a \in K\} \cup \{(a,0)|a \in K\}) = K^2 \setminus (V(x_1) \cup V(x_2)) = K^2 \setminus (V(x_1 \cdot x_2))$ when we associate x_1 with the first coordinate and x_2 with the second. Using 3.2.9 we obtain $\mathcal{O}(U_3) = K[x_1, x_2]_{x_1 \cdot x_2}$. □