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Wintersemester 2020/2021
2. December 2020

Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 4

Exercise 1.
Let X be a noetherian topological space and let $X = X_1 \cup \ldots \cup X_n$ be the decomposition into irreducible components. Show: If $U \subset X$ is a non-empty open subset, then the irreducible components of U are exactly the sets $X_i \cap U$ with $i = 1, \ldots, n$ for which $X_i \cap U \neq \emptyset$.

Proof: Let $U \subset X$ be a non-empty open subset and and let $X = X_1 \cup \ldots \cup X_n$ be the minimal decomposition into irreducible components.
Without restriction we assume that $U \cap X_i \neq \emptyset$ for $i = 1, \ldots, r$, with $r \leq n$ and $U \cap X_i = \emptyset$ for all $i \geq r+1$.
We want to prove that $U = (X_1 \cap U) \cup \ldots \cup (X_r \cap U)$ is a minimal irreducible decomposition of U. By the uniqueness of the minimal decomposition the claim then follows.
We start with showing: $X_i \cap U$ is irreducible in U for any $i = 1, \ldots, r$.
Assume $X_i \cap U$ is reducible in U. Then there exist $Y_1, Y_2 \subsetneq U$ closed, s.t. $Y_1, Y_2 \neq U \cap X_i$ and

$$X_i \cap U = Y_1 \cup Y_2 = (Z_1 \cap U) \cup (Z_2 \cap U)$$

where $Z_1, Z_2 \subset X$ closed with $Y_1 = Z_1 \cap U$, $Y_2 = Z_2 \cap U$.
Since X_i is irreducible and $X_i \cap U$ is non-empty and open in X_i, it follows that

$$\overline{Z_1 \cap U} \cup \overline{Z_2 \cap U} = \overline{U \cap X_i} = X_i.$$

Since X_i is irreducible and $\overline{Z_1 \cap U}$,$\overline{Z_2 \cap U}$ are closed in X_i (note that we do not need to distinguish between closed in X_i and closed in X here, because X_i is closed in X), we can assume without loss of generality that $X_i = \overline{Z_1 \cap U}$. It follows that

$$X_i \cap U = \overline{Z_1 \cap U} \cap U.$$

Now $\overline{Z_1 \cap U} \subset Z_1 \cap \overline{U}$, since Z_1 is closed in X. This implies

$$X_i \cap U = Z_1 \cap U = Y_1.$$

Contradiction to $X_i \cap U$ reducible in U.
Therefore $X_i \cap U$ is irreducible in U for all $i = 1, \ldots, r$.
Hence

$$U = (X_1 \cap U) \cup \ldots \cup (X_r \cap U)$$

is an irreducible decomposition of U. We need to show that it is minimal: If $X_i \cap U \subset X_j \cap U$ for some $i \neq j$, it follows that $\overline{X_i \cap U} \subset \overline{X_j \cap U}$. But since X_i and X_j are irreducible, $\overline{X_i \cap U} = X_i$ and $\overline{X_j \cap U} = X_j$, so $X_i \subset X_j$. **Contradiction.**
So the decomposition $U = (X_1 \cap U) \cup \ldots \cup (X_r \cap U)$ of U into irreducible components is minimal. □

Exercise 2. a) Let $\varphi : X \to Y$ be a continuous map of noetherian topological spaces and let $Z \subset Y$ be the closure of the image $\varphi(X)$. Prove: The maximal number of irreducible components in the minimal decomposition of Z is bounded by the number of irreducible components of the minimal decomposition of X.

Proof: Let $\varphi : X \to Y$ be a continuous map of noetherian topological spaces. Let $X = X_1 \cup \ldots \cup X_n$ be the minimal decomposition of X into irreducible components. By Prop. 1.6.3 $\varphi(X_i) \subset Z = \overline{\varphi(X)}$ is irreducible for each $i = 1, \ldots, n$.
So we can write $\varphi(X) = \varphi(X_1) \cup \ldots \cup \varphi(X_n)$. By exercise 3 of exercise sheet 2 it follows that

$$Z = \overline{\varphi(X)} = \overline{\varphi(X_1)} \cup \ldots \cup \overline{\varphi(X_n)}.$$

Lemma 1.5.3 states that the closure of an irreducible set is still irreducible. Since some of the $\overline{\varphi(X_i)}$ can be contained in each other (if φ is not injective) this decomposition is not necessarily minimal. But if follows that the maximal number of irreducible components of the minimal decomposition of Z is bounded by n. (This follows from the proof of Proposition 1.5.6) □

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b) Let $X = V(xy + x - y - 1) \subset \mathbb{C}^2$, $Y = \mathbb{C}$ and let $\varphi : X \to Y$ be defined by $\varphi((a, b)) = a$. Prove that X and Y are noetherian topological spaces, that φ is continuous with respect to the Zariski topology and that the number of irreducible components of the closure of $\varphi(X)$ is strictly smaller than the number of irreducible components of X.

Proof: Firstly we consider K^n for a field K. Then K^n is a noetherian topological space with the Zariski topology: Let $V_1 \supset V_2 \supset \ldots$ be a descending chain of Zariski closed subset. Then we consider the associated chain of vanishing ideals in $K[X_1, \ldots, X_n]$: $I(V_1) \subset I(V_2) \subset \ldots$. Since $K[X_1, \ldots, X_n]$ is noetherian, the ascending chain of ideals has to become stationary. Since the V_i are closed in the Zariski topology we have $V(I(V_i)) = V_i$. This implies that the descending chain of closed sets in K^n also becomes stationary. So K^n is a noetherian topological space.
By the above argument it follows immediately that $Y = \mathbb{C}$ is a noetherian topological space.
We know that $X \subset \mathbb{C}^2$ is closed in the Zariski topology. By Remark 1.5.5 and the above argument it follows that X is a noetherian topological space with the induced subspace topology.
The map $\varphi : X \to Y$ defined by $\varphi((a, b)) = a$ is polynomial in each coordinate entry of Y, so by Prop. 1.6.2 φ is continuous with respect to the Zariski topology.
It remains to determine the numbers of irreducible components of X and $\overline{\varphi(X)}$.
Now $X = V(xy + x - y - 1)$ is not irreducible itself: We write $xy + x - y - 1 = (x - 1)(y + 1)$ so $I(X) = \langle (x - 1)(y + 1)\rangle$ is not prime, since $(x - 1)(y - 1) \in I(X)$ but $x - 1$ and $y + 1 \notin I(X)$. Alternatively we can write $X = V(x - 1) \cup V(y - 1)$ and $V(x - 1), V(y + 1) \subsetneq X$ are both closed. Both $V(x - 1)$ and $V(y + 1)$ are irreducible since $\langle x - 1\rangle$ and $\langle y + 1\rangle$ are prime ideals (prop. 1.5.2). So the decomposition of X has two irreducible components.
We have $\overline{\varphi(X)} = Y$ since $\varphi|_{V(y+1)}$ is bijective onto Y, $\varphi(V(x - 1)) = 1 \in Y$ and Y is closed. As $Y = \mathbb{C}$ it is itself irreducible and therefore has only one irreducible component. □

Exercise 3.
Let K be a field and let $X := \{A \in \mathrm{Mat}(n \times n; K)|\ \mathrm{rank}(A) \leq 1\}$. Prove: X is irreducible in $\mathrm{Mat}(n \times n; K) = K^{n \times n}$.

Proof: We use Proposition 1.6.3, which states that the continuous image of an irreducible space is irreducible.
Let $A \in X$, then A is a square matrix of rank 1, so there is one row vector which generates the remaining $n - 1$ rows. Hence we can write A as a product of an n-dimensional column vector and an n-dimensional row vector. Since any such product yields a rank 1 $n \times n$-matrix, we can parametrise X via $\varphi : K^n \times K^n \to X$, $(v, w) \mapsto v \cdot w^t$, where $v, w \in K^n$ are column vectors. By the above arguments φ is surjective. Since any choice (v, w) determines $A = v \cdot w^t$ uniquely, φ is also injective.
X is a topological space with the subspace topology of the Zariski topology of $K^{n\times n}$.
$K^n \times K^n = K^{2n}$ is a topological space with the Zariski topology.
The mapping instruction is polynomial in each coordinate entry of X, so by Proposition 1.6.2 φ is continuous.
$K^n \times K^n = K^{2n}$ is irreducible by Proposition 1.5.2 since the coordinate ring of K^{2n} is $K[X_1, \ldots, X_{2n}]$ which is an integral domain.
Now we can apply Proposition 1.6.3 to $\varphi : K^{2n} \to X$. It follows that $X = \varphi(K^{2n})$ is irreducible. □.