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Introduction to Commutative Algebra and Algebraic Geometry
Solution to Exercise Sheet 5

Exercise 1.
Let $>$ be a monomial order on Mon_n. Prove that the following are equivalent:

1) $>$ is global.
2) $>$ is a well-ordering
3) $x_i > 1 \ \forall\, i = 1, \ldots, n$
4) $>$ refines $\geq_{\text{nat}}$, i.e. $\underline{x}^\alpha >_{\text{nat}} \underline{x}^\beta \ \Rightarrow \ \underline{x}^\alpha > \underline{x}^\beta$

Proof:

„1)⇒ 4)": Let $>$ be a global order and let $\underline{x}^\alpha, \underline{x}^\beta$ be monomials with $\underline{x}^\alpha >_{\text{nat}} \underline{x}^\beta$. It follows that $\alpha_i \geq \beta_i$ for all $i = 1, \ldots, n$. Therefore $\alpha_i - \beta_i \geq 1$ and thus, since $>$ is a global order, we have $\underline{x}^{\alpha-\beta} > 1$. By multiplying with $\underline{x}^\beta$ we obtain $\underline{x}^\alpha > \underline{x}^\beta$.

„4)⇒ 2)": A well-ordering is a total order for which every non-empty subset has a least element in this ordering. Let $M \subset \text{Mon}_n$ be an arbitrary subset. We need to show that M has a minimal element with respect to $>$. By Dickson's Lemma there exists a finite subset $B \subset M$ such that $\forall \underline{x}^\alpha \in M \ \exists \underline{x}^\beta \in B$ with $\underline{x}^\alpha \geqq_{\text{nat}} \underline{x}^\beta$. By 4) we know that for this finite subset $B \subset M$ we even have $\forall \underline{x}^\alpha \in M \ \exists \underline{x}^\beta \in B$ with $\underline{x}^\alpha > \underline{x}^\beta$. Therefore, we can conclude that the minimal element of M is contained in B. Since a monomial order is always a total order we can arrange the finitely many elements of B in ascending order and thus obtain the minimal element in M.

„2) ⇒ 3)": Assume there exists an i_0 with $x_{i_0} < 1$. It follows that $x_{i_0}^j > x_{i_0}^{j+1}$. We consider the subset

$$M = \{1, x_{i_0}, x_{i_0}^2, x_{i_0}^3, \ldots\} \subset \text{Mon}_n.$$

By our assumption this subset does not contain a minimal element. **Contradiction** to $>$ being a well-ordering. Therefore, $x_i > 1$ for all $i = 1, \ldots, n$.

„3)⇒ 1)": Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be arbitrary. By precondition: $x_i > 1$ for all $i = 1, \ldots, n$. It follows that $x_i^k > 1$ for all $k \in \mathbb{N}$. We write $\underline{x}^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Since $\alpha_i \in \mathbb{N}$ for all $i = 1, \ldots, n$ we have $x_i^{\alpha_i} > 1$ for all $i = 1, \ldots, n$. By repeatedly using transitivity we obtain $\underline{x}^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} > 1$, so $>$ is a global order.

Exercise 2. a) Prove that the set of monomials of $K[X]$ has exactly two orders, one of which is global.

Proof: The set of monomials of $K[X]$ is $\text{Mon}_1 = \{1, X, X^2, X^3, \ldots\}$. Since any monomial order is a total order any two elements of Mon_1 are comparable. Therefore, we need to decide for X and 1 between $X > 1$ and $X < 1$. It remains to show that this decision uniquely determines the monomial order.
Case 1: Since the monomial order is compatible with the semigroup structure it follows that $X > 1$ implies $X^{k+1} > X^k$ and more over by transitivity we have $X^k > X^m$ with $k > m$. So the choice $X > 1$ determines the order completely and this order is global.
Case 2: We have $X < 1$, so by the compatibility with the semigroup structure we have $X^{k+1} < X^k$ and, moreover, by transitivity we have $X^k < X^m$ with $k > m$. So the choice $X < 1$ determines the order completely and this order is not global. □

b) Prove that the set of monomials of $K[X, Y]$ has uncountable many orders.
Hint: Consider weighted degree reverse lexicographic orders.

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Proof: The set of monomials of $K[X,Y]$ is Mon_2.
We consider orders given by weighted degrees: Let $\alpha, \alpha' \in \mathbb{R}$ be two real numbers. We show that for $\alpha \neq \alpha'$ the two orders induced by $w_\alpha = (1,\alpha)$ and $w_{\alpha'} = (1,\alpha')$ are different.
W.l.o.g. we choose $\alpha < \alpha'$. Choose $\frac{p}{q} \in \mathbb{Q}$ with $\alpha < \frac{p}{q} < \alpha'$. It follows that $q\alpha < p < q\alpha'$. We now compare X^p with Y^q in the two orders induced by $w_\alpha = (1,\alpha)$ and $w_{\alpha'} = (1,\alpha')$: To compare X^p and Y^q with respect to $w_\alpha = (1,\alpha)$ we compute $(p,0)\cdot(1,\alpha) = p$ and $(0,q)\cdot(1,\alpha) = \alpha\cdot q < p$. It follows that $X^p >_{w_\alpha} Y^q$.
For $w_{\alpha'} = (1,\alpha')$ we compute $(p,0)\cdot(1,\alpha') = p$ and $(0,q)\cdot(1,\alpha') = \alpha'\cdot q > p$. It follows that $X^p <_{w_{\alpha'}} Y^q$.
So for two real numbers the induced orders by the weighted degrees are different, so every real number gives a different order. Hence, there are uncountable many orders on Mon_2. □

Exercise 3.
Let K be an algebraically closed field. Let $A \subset K^n$ be an affine varieties and let $f : A \to K^m$, $a \mapsto (f_1(a),\dots,f_m(a))$ be a polynomial map, so $f_1,\dots,f_m \in K[X_1,\dots,X_n]$. We write $K[X_1,...,X_n]$ for the coordinate ring to K^n and $K[Y_1,...,Y_m]$ for the coordinate ring to K^m.
Prove: $\overline{f(A)} = V(\langle I(A), Y_1 - f_1,\dots,Y_m - f_m\rangle_{K[X_1,\dots,X_n,Y_1,\dots,Y_m]} \cap K[\underline{Y}])$.

Proof: Let $\Gamma_f = \{(a,f(a))|a \in A\} \subset A \times K^m$ be the graph of f and let $\pi_2 : A\times K^m \to K^m$ be the projection to K^m. By Prop. 1.6.8 we know that Γ_f is an affine variety and by Prop. 1.6.4 we know that π_2 is a polynomial map. Since $f(A) = \pi_2(\Gamma_f)$, we know:

$$\begin{aligned}\overline{f(A)} &= \overline{\pi_2(\Gamma_f)}\\ \overset{\text{Prop. 1.5.14}}&{=} V(I(\Gamma_f)\cap K[\underline{Y}])\end{aligned}$$

where $I(\Gamma_f)$ is an ideal in $K[X_1,\dots,X_n,Y_1,\dots,Y_m]$.
However, we do not need to compute $I(\Gamma_f)$. For the application of 1.5.14 it suffices to find an ideal I such that $\Gamma_f = V(I)$.
Let $\pi_1 : A\times K^m \to A$ be the projection to A and let $g : K^m \to K^m$ be the identity with $g = (g_1,\dots,g_m)$. Since f is a polynomial map defined on $A \subset K^n$ we can consider it as the restriction of a polynomial map $K^n \to K^m$. We denote this map also by f.

$$\begin{aligned}\Gamma_f &= (A\times K^m)\cap\{(a,b)\in K^n\times K^m | g(b) = f(a)\}\\ \overset{\text{Prop. 1.6.9}}&{=} (A\times K^m)\cap(K^n\times K^m)\cap V(f_1\circ\pi_1 - g_1\circ\pi_2,\dots,f_m\circ\pi_1 - g_m\circ\pi_2)\\ &= (A\times K^m)\cap V(f_1\circ\pi_1 - g_1\circ\pi_2,\dots,f_m\circ\pi_1 - g_m\circ\pi_2)\end{aligned}$$

By the proof of 1.6.4 we know $I(A\times K^m) = \langle I(A)\rangle_{K[\underline{X},\underline{Y}]}$. It follows that

$$\begin{aligned}\Gamma_f &= V(I((A\times K^m)\cap V(f_1\circ\pi_1 - g_1\circ\pi_2,\dots,f_m\circ\pi_1 - g_m\circ\pi_2)))\\ &= V(\langle I(A), f_1\circ\pi_1 - g_1\circ\pi_2,\dots,f_m\circ\pi_1 - g_m\circ\pi_2\rangle_{K[\underline{X},\underline{Y}]})\end{aligned}$$

Let $I = \langle I(A), f_1\circ\pi_1 - g_1\circ\pi_2,\dots,f_m\circ\pi_1 - g_m\circ\pi_2\rangle_{K[\underline{X},\underline{Y}]}$ with $I(A) \subset K[\underline{X}]$. It follows that

$$\begin{aligned}\overline{f(A)} &= \overline{\pi_2(\Gamma_f)}\\ \overset{\text{Prop. 1.5.14}}&{=} V(I\cap K[\underline{Y}])\\ &= V(\langle I(A), f_1\circ\pi_1 - g_1\circ\pi_2,\dots,f_m\circ\pi_1 - g_m\circ\pi_2\rangle_{K[X_1,\dots,X_n,Y_1,\dots,Y_m]}\cap K[\underline{Y}])\end{aligned}$$

Here we have used that $f_1,\dots,f_m \in K[\underline{X}]$ so that $f_i\circ\pi_1 = f_i$ and that the equations are coordinate-wise, so that $f_i\circ\pi_1 - g_i\circ\pi_2 = f_i - Y_i$. □