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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 10

Exercise 1.
Let X be an affine variety and let $Y \subset X$ be closed. Endow Y with the subspace topology. Show: The inclusion map $i : Y \hookrightarrow X$ is a morphism of spaces with functions, $i^*(\mathcal{O}_X(X)) = \mathcal{O}_Y(Y)$ and $\ker(i^*) = I_X(Y)$.

This finishes the proof of Lemma 3.3.8.

Solution: The inclusion map is the identity on each coordinate, thus it is a continuous map. Let $V \subset X$ be open, $h \in \mathcal{O}_X(V)$. We have to show that $h \circ i \in \mathcal{O}_Y(i^{-1}(V))$.
We know $i^{-1}(V) = V \cap Y$ and $h \circ i = h_{|_Y}$.
Since Y is endowed with the subspace topology, and $V \subset X$ open, we know that $V \cap Y$ is open. Moreover, we know that h is regular on V, therefore h is regular on $V \cap Y \subset V$ and so $h_{|_Y}$ is regular on $V \cap Y$.

The remaining claims of Lemma 3.3.8:

$$i^*(\mathcal{O}_X(X)) = \mathcal{O}_Y(Y)$$

This is true, since i is injective, and since $i \mapsto i^*$ is a contravariant functor, i^* is surjective.

$$\ker(i^*) = I_X(Y)$$

This holds, because

$$\begin{aligned} f \in \ker(i^*) &\Leftrightarrow f \circ i = 0 \\ &\Leftrightarrow f_{|_Y} = 0 \\ &\Leftrightarrow f_{|_Y} \text{ is in the vanishing ideal of Y in X} \\ &\Leftrightarrow f_{|_Y} \in I_X(Y) \subset K[X] = \mathcal{O}_X(X). \end{aligned}$$

□

Exercise 2.
Let K be an algebraically closed field.
Prove: The line L described by $3x + 4y = 7$ in $\mathbb{K}^2$ is isomorphic to $\mathbb{K}^1$.

Solution: We know $K[L] \cong K[x, y]/\langle 3x+4y-7\rangle$. Furthermore, we see that $K[x, y]/\langle 3x+4y-7\rangle \cong K[x] \cong K[\mathbb{A}^1_K]$ by considering the map $K[x, y] \to K[x]$, $f(x, y) \mapsto f(x, \frac{(-3x+7)}{4})$ which is surjective and satisfies $\langle 3x+4y-7\rangle \subset \ker$ (since $\overline{y} = \overline{\frac{(-3x+7)}{4}}$). Now $\langle 3x+4y-7\rangle \supset \ker$, since for $f \in \ker$ we know that $f_{|_L}$ vanishes, so $L \subset V(f)$ which implies that $f \in \langle 3x + 4y - 7\rangle$. So we can apply the homomorphism theorem and obtain $K[L] \cong K[x, y]/\langle 3x + 4y - 7\rangle \cong K[x] \cong K[\mathbb{A}^1_K]$.
That choice of congruence means that we've chosen the isomorphism $\mathbb{A}^1 \to L$ given by $(x) \mapsto (x, \frac{(-3x+7)}{4})$, or the reverse morphism $L \to \mathbb{A}^1$ given by $(x, y) \mapsto x$. □