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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 11

Exercise 1.
Let K be an algebraically closed field.

a) Let $g \in K[x_1, \dots, x_n] \setminus \{0\}$. Prove: $\dim(D(g)) = n$.

Solution: We know from the proof of 3.3.10 that $D(g) \cong V(gx_{n+1} - 1) \subset K^{n+1}$ is a hypersurface ($g \neq 0$), and by 4.1.12 it is purely $(n+1) - 1 = n$-dimensional. □

b) Let X be an irreducible affine variety in K^n and let $f \in \mathcal{O}(X)$, $f \neq 0$.
Prove: $\dim(X_f) = \dim(X)$. (Recall: $X_f = X \cap D(f)$.)

Solution: It follows from the proof of 3.3.10 that $X_f \cong V(I, fx_{n+1} - 1)$, where $X = V(I)$. Now

$$\begin{aligned} K[X_f] &= K[x_1, \dots, x_{n+1}]/\langle I, fx_{n+1} - 1\rangle \\ &= K[x_1, ..., x_n]_f/\langle I\rangle_f \\ &= (K[x_1, ..., x_n]/I)_f \\ &= K[X]_f \end{aligned}$$

Sine $f \in \mathcal{O}(X) = K[X]$ taking the quotientfield of $K[X]_f$ gives the same result as taking the quotient field of $K[X]$:

$$\mathsf{Quot}(K[X]_f) = K(X) = \mathsf{Quot}(K[X])$$

So $\dim(X_f) = \mathsf{trdeg}_K(K(X_f)) = \mathsf{trdeg}_K(K(X)) = \dim(X)$. □