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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 13

Exercise 1.
Let K be an algebraically closed field. The morphism $\varphi : K \to K$, $z \mapsto z^2$ maps a one dimensional affine variety to itself. Prove, that φ is not birational.

Solution: If φ is birational, φ^* has to be extendable to the fraction fields and has to be an isomorphism of the fraction fields.
K is an irreducible variety and moreover we see that $\varphi : K \to K$ is surjective, since K is algebraically closed. Thus, we can extend $\varphi^* : K[x] \to K[z]$, $x \to z^2$ to the fraction fields:

$$\varphi^* : K(x) \to K(z) \;\; x \to z^2.$$

Now, φ^* is not bijective, since $z \notin \mathrm{im}(\varphi^*) = \{\frac{f}{g} | f, g \in K[z]$ with only even exponents $\} = \{\frac{f(z^2)}{g(z^2)} | f, g \in K[z]\}$. □