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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 5

Exercise 1.
Let $f = x^2y^3 - z + 3x^2z + xy - 10y^2z + xz^2, g = x^4z^2 + 7y^3z + z^2 - 3xy + 5 \in K[x, y, z]$. Write f, g in the

a) lexicographic order with $z > y > x$.

b) degree reverse lexicographic order with $y > z > x$.

c) the weighted degree reverse lexicographic order given by $w = (2, -1, 1)$ for which the first coordinate corresponds to x, the second to y and the third to z (i.e. $x_1 = x, x_2 = y, x_3 = z$).

Solution:

a) We have to order the monomials appearing in f by the lexicographic order with $z > y > x$:

$$z^2x >_{lp} zy^2 >_{lp} zx^2 >_{lp} z >_{lp} y^3x^2 >_{lp} yx$$

so $f = z^2x - 10y^2z + 3zx^2 - z + y^3x^2 + yx$. For g we obtain

$$z^2x^4 >_{lp} z^2 >_{lp} zy^3 >_{lp} yx >_{lp} 1$$

so $g = z^2x^4 + z^2 + 7zy^3 - 3yx + 5$.

b) We have to order the monomials appearing in f by the reverse lexicographic order with $y > z > x$:

$$y^3x^2 >_{dp} y^2z >_{dp} z^2x >_{dp} zx^2 >_{dp} yx >_{dp} z$$

so $f = y^3x^2 - 10y^2z + z^2x + 3zx^2 + yx - z$. For g we obtain

$$z^2x^4 >_{dp} y^3z >_{dp} z^2 >_{dp} yx >_{dp} 1$$

so $g = z^2x^4 + 7zy^3 + z^2 - 3yx + 5$.

c) We have to order the monomials appearing in f by the weighted degree reverse lexicographic order given by $w = (2, -1, 1)$ for which the first coordinate corresponds to x, the second to y and the third to z:

$$\begin{aligned}
\deg_w(xz^2) &= 2 + 2 = 4 \\
\deg_w(y^2z) &= -2 + 1 = -1 \\
\deg_w(x^2z) &= 4 + 1 = 5 \\
\deg_w(z) &= 1 \\
\deg_w(x^2y^3) &= 4 - 3 = 1 \\
\deg_w(xy) &= 2 - 1 = 1
\end{aligned}$$

For the monomials of the same degree we get with the weighted degree reverse lexicographic ordering: $xy >_{wp(w)} x^2y^3 >_{wp(w)} z$. So it follows that

$$x^2z >_{wp(w)} xz^2 >_{wp(w)} xy >_{wp(w)} x^2y^3 >_{wp(w)} z >_{wp(w)} y^2z$$

so $f = 3zx^2 + z^2x + yx + y^3x^2 - z - 10y^2z$. For g we compute:

$$\begin{aligned}
\deg_w(x^4z^2) &= 8 + 2 = 10 \\
\deg_w(y^3z) &= -3 + 1 = -2 \\
\deg_w(z^2) &= 2 \\
\deg_w(xy) &= 2 - 1 = 1
\end{aligned}$$

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So it follows that

$$x^4z^2 >_{wp(w)} z^2 >_{wp(w)} xy >_{wp(w)} y^3z$$

so $g = x^4z^2 + z^2 - 3xy + 5 + 7y^3z$.

Exercise 2.
Matrices $A \in \mathsf{GL}(n, \mathbb{R})$ with real entries can be used to obtain a monomial ordering on Mon_n by setting

$$x^\alpha >_A x^\beta :\Leftrightarrow A\alpha > A\beta$$

where $>$ on the right hand side is the lexicographical ordering on $\mathbb{R}^n$.
Find the matrices that induce the lexicographic and the degree reverse lexicographic ordering on Mon_n with $X_1 > \ldots > X_n$.

Solution: The lexicographic ordering $>$ on $\mathbb{R}^n$ means that $\alpha > \beta \Leftrightarrow \exists\, k$ s.t. $\alpha_1 = \beta_1, \ldots \alpha_{k-1} = \beta_{k-1}$ and $\alpha_k > \beta_k$.
It follows that the matrix ordering induced by $A = \mathbf{1}_n$ induces the lexicographic ordering.

For the degree reverse lexicographic ordering we need a matrix A such that $\underline{x}^\alpha >_{dp} \underline{x}^\beta$ if and only if $A\alpha > A\beta$ in the lexicographic ordering of $\mathbb{R}^n$.
We have $\underline{x}^\alpha >_{dp} \underline{x}^\beta$ if and only if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and there exists k s.t. $\alpha_n = \beta_n, \ldots \alpha_{k+1} = \beta_{k+1}$ and $\alpha_k < \beta_k$.
We observe, that we never need to compare the first entries of α and β, since if $|\alpha| = |\beta|$ and $\alpha_n = \beta_n, \ldots, \alpha_2 = \beta_2$ it also implies that $\alpha_1 = \beta_1$.
So to compare in the lexicographic order the vector $A\alpha$ needs to contain $|\alpha|$ as first entry and needs to filled with $-\alpha_n, \ldots, -\alpha_2$, since the degree reverse lexicographic order $>_{dp}$ means that if $|\alpha| = |\beta|$ and there exists k s.t. $\alpha_n = \beta_n, \ldots \alpha_{k+1} = \beta_{k+1}$ and $\alpha_k < \beta_k$ we have $\underline{x}^\alpha >_{dp} \underline{x}^\beta$. Therefore the matrix A we are looking for has an anti-diagonal filled with -1 while the 1-th row to the matrix is filled with ones:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & -1 & \cdots & 0 & 0 \end{pmatrix} \in \mathsf{GL}(n, K).$$