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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 6

Exercise 1.
For $R = \mathbb{Q}[x, y, z]$ compute spoly$(xy - y, 2x^2 + yz)$ and spoly$(xy - y, -y^3 - 2y)$ with respect to $>_{lp}$.

Solution:

- LM$(2x^2 + yz) = x^2$ and LM$(xy - y) = xy$ so $gcd(x^2, xy) = x$.

$$\begin{aligned}\text{spoly}(xy - y, 2x^2 + yz) &= \frac{2x^2}{x} \cdot (xy - y) - \frac{xy}{x} \cdot (2x^2 + yz)\\ &= 2x \cdot (xy - y) - y \cdot (2x^2 + yz)\\ &= -2xy - y^2 z\end{aligned}$$

- LM$(-y^3 - 2y) = y^3$ and LM$(xy - y) = xy$ so $gcd(y^3, xy) = y$.

$$\begin{aligned}\text{spoly}(xy - y, -y^3 - 2y) &= \frac{-y^3}{y} \cdot (xy - y) - \frac{xy}{y} \cdot (-y^3 - 2y)\\ &= -y^2 \cdot (xy - y) - x \cdot (-y^3 - 2y)\\ &= y^3 + 2xy\end{aligned}$$

Exercise 2.
A polynomial $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \underline{x}^\alpha \in K[x_1, \ldots, x_n]$ is called *homogeneous* if for all α with $a_\alpha \neq 0$ the absolute value $|\alpha|$ is constant.
An ideal $I \subset K[x_1, \ldots, x_n]$ is called *homogeneous* if it is generated by a set of homogeneous polynomials.

Let I be a homogeneous ideal in $K[\underline{x}]$. Show that the degree reverse lexicographic ordering $>_{dp}$ satisfies the property:

$$L(I + \langle x_n^d \rangle) = L(I) + \langle x_n^d \rangle \text{ for any } d \geq 1.$$

Hint: Show that if I is homogeneous, we only have to consider homogeneous polynomials, e.g. $L(I) = \langle \text{LM}(f) | f \in I, f \text{ homogeneous} \rangle$.
You are allowed to use that the sum of homogeneous ideals is again a homogeneous ideal.

Proof: First we prove the statement in the hint: For every $f \in I$ homogeneous we know that LM(f) with respect to $>_{dp}$ is contained in the leading ideal $L_{>_{dp}}(I)$ w.r.t. $>_{dp}$.
For the other inclusion let $f \in I$. We write $g = \deg(\text{LM}(f))$-part of $f \in K[\underline{x}]$. Since I is homogeneous we obtain that $g \in I$. Moreover, we have LM$(f) =$ LM(g) with $g \in I$ homogeneous.
Now we prove the statement from the exercise:
„$\supset$“: $L_{>_{dp}}(I) \subset L_{>_{dp}}(I + \langle x_n^d \rangle)$ and $\langle x_n^d \rangle \subset L_{>_{dp}}(I + \langle x_n^d \rangle)$
„$\subset$“: Let $f = g + x_n^d \cdot h$, $g \in I, h \in K[\underline{x}]$. Since $I + \langle x_n^d \rangle$ is homogeneous (Hint) we can assume that f is homogeneous.
Note that LM(f) appears in g or in $x_n^d \cdot h$. We distinguish these cases:

- LM(f) appears in $x_n^d \cdot h \Rightarrow$ LM$(f) \in x_n^d \cdot h$
- LM(f) appears in g:
  If LM$(f) =$ LM$(g) \Rightarrow$ LM$(f) \in L(I)$ since $g \in I$.
  Otherwise LM$(g) >$ LM(f) and $\underline{x}^\gamma :=$ LM(f) appears in $g_\gamma = \deg(\text{LM}(f))$-part of g. Further we know that LM$(g) = x_n^d \cdot h'$ for some $h' \in K[\underline{x}]$, since $f = g + x_n^d \cdot h$. Since $\underline{x}^\beta := x_n^d \cdot h' =$ LM$(g) >_{dp} \underline{x}^\gamma$ in $>_{dp}$ we have $\beta_n < \gamma_n$, so $\underline{x}^\gamma \in \langle x_n^d \rangle$. □