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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 7

Exercise 1.
Let $R = \mathbb{Q}[x, y, z]$ and $I = \langle xy - y, 2x^2 + yz, y - z\rangle \subset R$. Compute whether $f = xz^3 - 2y^2$ belongs to I.

Solution: We know from Exercise 2 Sheet 7 that $S = \{xy - y, 2x^2 + yz, y - z, -y^3 - 2y\}$ is a Groebner basis of I with respect to $>_{lp}$ where $x > z > y$. We need to apply Proposition 2.3.2 to f with respect to S. We use Buchberger ID for the indeterminate division of f with S.
Write $f_1 = xy - y, f_2 = 2x^2 + yz, f_3 = y - z, f_4 = -y^3 - 2y$.
1. Step: We know $\mathsf{LM}(f) = xz^3$. This gets divided by $\mathsf{LM}(f_3) = z$. Set:

$$q_3 = \frac{\mathsf{LT}(f)}{\mathsf{LT}(f_3)} = -xz^2$$

and

$$\begin{aligned} r &= \frac{\mathsf{spoly}(f, f_3)}{\mathsf{LC}(f_3)} \\ &= f - \frac{\mathsf{LT}(f)}{\mathsf{LT}(f_3)} f_3 \\ &= f + xz^2 f_3 \\ \\ &= -2y^2 + xyz^2 \end{aligned}$$

2. Step: We know $\mathsf{LM}(r) = xyz^2$. This gets divided by $\mathsf{LM}(f_3) = z$. Set:

$$q_3 = -xz^2 + \frac{\mathsf{LT}(r)}{\mathsf{LT}(f_3)} = -xz^2 - xyz$$

and

$$\begin{aligned} r &= \frac{\mathsf{spoly}(r, f_3)}{\mathsf{LC}(f_3)} \\ &= -2y^2 + xyz^2 - \frac{\mathsf{LT}(-2y^2 + xyz^2)}{\mathsf{LT}(f_3)} f_3 \\ &= -2y^2 + xyz^2 - \frac{xyz^2}{-z}(y - z) \\ &= -2y^2 + xyz^2 + xyz(y - z) \\ \\ &= -2y^2 + xy^2 z \end{aligned}$$

3. Step: We know $\mathsf{LM}(r) = xy^2 z$. This gets divided by $\mathsf{LM}(f_3) = z$. Set:

$$q_3 = -xz^2 - xyz + \frac{\mathsf{LT}(r)}{\mathsf{LT}(f_3)} = -xz^2 - xyz - xy^2$$

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and

$$\begin{aligned}
r &= \frac{\mathsf{spoly}(r, f_3)}{\mathsf{LC}(f_3)} \\
&= -2y^2 + xy^2z - \frac{\mathsf{LT}(-2y^2 + xy^2z)}{\mathsf{LT}(f_3)} f_3 \\
&= -2y^2 + xy^2z - \frac{xy^2z}{-z}(y - z) \\
&= -2y^2 + xy^2z + xy^2(y - z) \\
&= -2y^2 + xy^3
\end{aligned}$$

4. Step: We know $\mathsf{LM}(r) = xy^3$. This gets divided by $\mathsf{LM}(f_4) = y^3$. Set:

$$q_4 = \frac{\mathsf{LT}(r)}{\mathsf{LT}(f_4)} = -x$$

and

$$\begin{aligned}
r &= \frac{\mathsf{spoly}(r, f_4)}{\mathsf{LC}(f_4)} \\
&= -2y^2 + xy^3 - \frac{\mathsf{LT}(-2y^2 + xy^3)}{\mathsf{LT}(f_4)} f_4 \\
&= -2y^2 + xy^3 - \frac{xy^3}{-y^3}(-y^3 - 2y) \\
&= -2y^2 + xy^3 + x(-y^3 - 2y) \\
&= -2y^2 - 2xy
\end{aligned}$$

5. Step: We know $\mathsf{LM}(r) = xy$. This gets divided by $\mathsf{LM}(f_1) = xy$. Set:

$$q_1 = \frac{\mathsf{LT}(r)}{\mathsf{LT}(f_1)} = -2$$

and

$$\begin{aligned}
r &= \frac{\mathsf{spoly}(r, f_1)}{\mathsf{LC}(f_1)} \\
&= -2y^2 - 2xy - \frac{\mathsf{LT}(-2y^2 - 2xy)}{\mathsf{LT}(f_1)} f_1 \\
&= -2y^2 - 2xy - \frac{-2xy}{xy}(xy - y) \\
&= -2y^2 - 2xy + 2(xy - y) \\
&= -2y^2 - 2y
\end{aligned}$$

Now $\mathsf{LM}(-2y^2 - 2y) = y^2$ is not divided by any of the leading monomials of the f_i since $\mathsf{LM}(f_1) = xy$, $\mathsf{LM}(f_2) = x^2$, $\mathsf{LM}(f_3) = z$, $\mathsf{LM}(f_4) = y^3$. So the algorithm terminates.
We have $f = (-2)(xy - y) + (-xz^2 - xyz - xy^2)(y - z) - x(-y^3 - 2y) + (-2y^2 - 2y)$.

Since $r = -2y^2 - 2y \neq 0$ we have $f \notin I$. □